Tracking Control Design for Discrete-time Polynomial Systems: A Sum-of-Squares Approach

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Abstract—In this paper, tracking control synthesis problem for nonlinear polynomial discrete-time systems are studied. Proposed controller drives the plant such that the state vector of the plant follows those of a stable reference model. The objective is to design a controller such that the energy gains from the exogenous signals that are the reference signal and the state vector of the reference model, to the tracking error to be less or equal to prescribe thresholds. The main difficulty in the problem of designing tracking nonlinear discrete-time control law for the polynomial discrete time systems is that in general this problem may not be formulated as a convex problem. With proper selection of Lyapunov function and based on Lyapunov theory and by using sum of square approach, sufficient conditions for existence of controller are presented in terms of a feasibility SOS programming problem that can be solved using numerical solvers such as SOSTOOLS. Finally, the performance of proposed approach will be shown using the simulation of several examples.

Index Terms— Nonlinear control, polynomial discrete-time systems, tracking control, sum of square (SOS)

I. INTRODUCTION

In the recent decade stability analysis and controller synthesis problems for a class of nonlinear systems known as polynomial systems have mentioned by many researchers. In the aforementioned class of systems, equations describing the dynamics of system are polynomial functions of its states. The development of sum of square (SOS) methods in the analysis and synthesis of nonlinear control systems is one of the main reasons for attention to the polynomial systems [1-3]. Giving implicit conditions based on positive polynomial formulation that converts the controller synthesis problem to a feasibility SOS programming problem and can be solved with some third party toolboxes such as SOSTOOLS [4], YALMIP [5], GloptiPoly [6] is one of the methods using SOS for system analysis and synthesis

Studies on polynomial systems are seen in both categories of continuous-time and discrete-time systems. In each of categories, controller syntheses for stabilization and tracking control problems have been addressed by researchers. There are many researches in the literature that have dealt with the stabilization of continuous-time polynomial systems. For example see [7-18]. In comparison with the studies on the stabilization of continuous time polynomial systems, fewer researches found in the literature that have addressed the tracking problem for this class of systems. For example see [19].

On the other hand, some researchers have addressed controller synthesis for discrete-time polynomial systems [20-24]. As mentioned earlier, giving conditions for controller synthesis as SOS constraints is the main challenge that the researchers are faced. As one of the first research, it can be pointed to [20]. In [20] authors have chosen a special form for system and Lyapunov function. With that selection, the conditions for controller synthesis are obtained as SOS constraints. In [21] by selecting the system in a general form, the synthesis of stabilizing polynomial H_∞ state feedback controller has addressed. The authors have selected Lyapunov function similar to quadratic form with a state dependent Lyapunov matrix. It has resulted the emergence of non-convex terms in conditions presented for controller synthesis. To fix it, with assumption on the size of appeared non-convex terms, controller gains have selected so that the non-convex terms tend toward zero. Some authors tried to avoid from non-convex terms by selecting specific structures for controller. For example in [22] authors have selected a controller with integrator. In that paper the controller synthesis conditions for a class of system with norm bounded uncertainty are obtained. Based on the method presented in [22] the author in [23] and [24] have been designed nonlinear H_∞ control with integral structure for polynomial discrete-time systems.

In this paper, the tracking control synthesis problem for discrete-time polynomial system will be addressed. The sufficient conditions for controller synthesis are given in terms of polynomial matrix inequalities, which are formulated as SOS constraints in a feasibility semi-definite programming problem. The merit of the proposed method will be shown using the simulation of some examples using SOSTOOLS.

The paper is organized as follows: Section 2 will introduce some preliminaries and discrete-time polynomial model for system and controller. In Section 3, main results for the synthesis of discrete-time polynomial feedback controller will

be introduced. The simulations of several examples are given in Section 4 to show the merits of the proposed approach. Finally, the conclusions of the paper are presented in Section 5.

II. NOTATIONS AND PRELIMINARIES

A. Notations

Let $\mathbb{R}$ denotes the set of real number and $\mathbb{R}^n$ denotes the set of real space with dimension n. A matrix $M \prec 0$ means that the matrix M is a negative definite matrix. $x(k) \in \mathbb{R}^n$ denotes the state vector of a dynamical system where k is the discrete-time index $.\tilde{x} = \left[x_{k_1}, \ldots, x_{k_q} \right]^T$ denotes a vector containing those state of system that the input does not affect on them directly. In other words the entire elements of corresponding rows in the input matrix of the system are zeros. The notation $diag(A_1, \ldots, A_n)$ denotes a diagonal block matrix.

B. Preliminaries

Definition 2.1 [1]: A multivariate polynomial $f(x)$, $x \in \mathbb{R}^n$, is a sum of squares if there exist polynomials $f_i(x), i = 1, \ldots, m$ such that

$$f(x) = \sum_{i=1}^m f_i^2(x) \qquad (1)$$

If a decomposition for $f(x)$ in the form of (1) can be obtained, it is clear that $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Proposition 2.1 [1]: Let $f(x)$ be a polynomial in $x \in \mathbb{R}^n$ of degree $2d$. Let $\hat{x}(x)$ be a column vector whose entries are all monomials in x with degree smaller than d. A monomial in $x(t)$ is a function of the form $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$, are nonnegative integers and $d = \alpha_1 + \cdots + \alpha_n$. Therefore, $f(x)$ is said to be an SOS if and only if there exists a positive semi-definite matrix Q such that:

$$f(x) = \hat{x}(x)^T Q \hat{x}(x) \qquad (2)$$

Proposition 2.2 [7]: Let $F(x) \succcurlyeq 0$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. Furthermore, let $\hat{x}(x)$ denotes a column vector whose entries are all monomials in $x(t)$ with degree no greater than d, and consider the following conditions:

1- $F(x) \succcurlyeq 0$ for all $x \in \mathbb{R}^n$

2- $v^T F(x) v$ is an SOS, where $v \in \mathbb{R}^n$

3-There exists a positive semi-definite matrix Q such that:
$v^T F(x) v = (v \otimes \hat{x}(x))^T Q (v \otimes \hat{x}(x))$ where $\otimes$ denotes the Kronecker product.

Then $1 \Leftarrow 2, 2 \Leftrightarrow 3$.

Proposition 2.3(Congruence transformation): Consider M is a square matrix and T is a nonsingular one. Hence $M \prec 0$ if and only if $T^* M T \prec 0$; where $*$ denotes complex conjugate transpose of T.

Lemma 2.1 [25]. (Schur Complement): Let $A(x) = A^T(x)$ and $C(x) = C^T(x)$ and $D(x)$ depend affinely on x. Then

$$\begin{bmatrix} A(x) & D(x) \\ D^T(x) & C(x) \end{bmatrix} \succ 0$$

Is equivalent to

$$C(x) \succ 0, \quad A(x) - D(x) C^{-1}(x) D^T(x) \succ 0$$

or

$$A(x) \succ 0, \quad C(x) - D^T(x) A^{-1}(x) D(x) \succ 0$$

Lemma 2.2: The following statement holds:

$$\begin{bmatrix} A - B\,D^{-1}B^T & L \\ L^T & M \end{bmatrix} \succ 0 \Leftrightarrow \begin{bmatrix} A & B & L \\ B^T & D & 0 \\ L^T & 0 & M \end{bmatrix} \succ 0$$

Proof: Start from the left hand side and apply Schur complement on it; that results $A - BD^{-1}B^T - LM^{-1}L^T \succ 0 \to A - \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} D^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} B^T \\ L^T \end{bmatrix} \succ 0$. By appling Schur complement on the last inequality it can conclude the other side.□

C. Discrete-time polynomial plant

Consider the plant as

$$\begin{aligned} x(k+1) &= A(x(k))x(k) + B(x(k))u(k) \\ y(k) &= Cx(k) \end{aligned} \qquad (3)$$

where $A(x(k)) \in \mathbb{R}^{n \times n}$, $B(x(k)) \in \mathbb{R}^{n \times m}$ are the system and input matrices, respectively. $u(k) \in \mathbb{R}^m$ is the input vector, $y(k) \in \mathbb{R}^l$ is the output vector, $x(k) \in \mathbb{R}^n$ is the system state vector and $C \in \mathbb{R}^{l \times n}$ is a constant matrix.

To facilitate the stability analysis, like that is done in [7], and to avoid appearing non-convex conditions, it is assumed that the state vector of system can be partitioned as $x(k) = [\xi(k) \quad \tilde{x}(k)]^T$ where $\tilde{x}(k)$ denotes the states that does not affect directly from input signal. Therefore the subsystem that does not affected directly from input is shown as

$$\tilde{x}(k+1) = \tilde{A}(x(k))x(k) \qquad (4)$$

where $\tilde{A}(x(k)) \in \mathbb{R}^{q \times n}$.

D. Reference model

Consider a stable reference model as

$$\begin{aligned} x_r(k+1) &= A_r x_r(k) + B_r r(k) \\ y_r(k) &= C x_r(k) \end{aligned} \qquad (5)$$

where $x_r(k) \in \mathbb{R}^n$ is the state vector, $A_r \in \mathbb{R}^{n \times n}$ and $B_r \in \mathbb{R}^{n \times m}$ are the system and input matrices of the reference model, respectively. $r(k) \in \mathbb{R}^m$ is the desired input vector and $y_r(k) \in \mathbb{R}^l$ is the output vector of the reference model.

E. Discrete-time polynomial controller

Consider the feedback polynomial controller as:

$$u(k) = F(h)e_y(k) + G(h)y_r(k) \qquad (6)$$

where $F(h) \in \mathbb{R}^{m \times l}$ and $G(h) \in \mathbb{R}^{m \times l}$ are controller gains. h is a vector as $h = [y^T(k) \quad y_r^T(k)]^T$. In other words, the controller gains depend on the system and reference model outputs.

III. CONTROLLER DESIGN

Consider the discrete-time system (3). Substituting control law (6) in to (3) results that the closed-loop system can be represented as:

$$\begin{aligned} x(k+1) = {}& A(x(k))x(k) \\ & + B(x(k))\Big(F(h)e_y(k) \\ & + G(h)y_r(k) \Big) \end{aligned} \qquad (7)$$

The objective is to find controller gains $F(h)$ and $G(h)$ such that the closed-loop system isstable and the output vector of the closed-loop systems followsthe output vector of the reference model.

In the following theorem, we present necessary conditions for the existence of the tracking control law.

Theorem 1: The feedback polynomial controller (6) drives the states of the system (3) to follow those of the stable reference model (5) if there exists pre-defined scalar polynomial functions $\varepsilon_1(x) > 0$ and $\varepsilon_2(x) \geq 0$, decision polynomial matrix $X(\tilde{x}) = X(\tilde{x})^T \in \mathbb{R}^{n \times n}$ and decision polynomial matrices $M(h) \in \mathbb{R}^{m \times n}$ and $G(h) \in \mathbb{R}^{m \times l}$, such that:

$$\begin{aligned} v^T\big(X(\tilde{x}) - \varepsilon_1(x)\big)v &\quad \text{is SOS} \\ -\bar{\rho}^T\big(\Omega(x) + \varepsilon_2(x)\big)\bar{\rho} &\quad \text{is SOS} \end{aligned} \tag{8}$$

where

$$\Omega(x)$$
$$= \begin{bmatrix} X & X & 0 & 0 & * \\ X & Q^{-1} & 0 & 0 & * \\ 0 & 0 & \sigma_1^2 I & 0 & * \\ 0 & 0 & 0 & \sigma_2^2 I & * \\ A(x)X + B(x)M(x) & 0 & ((A(x)-A_r)+B(x)G(h)C) & -B_r & X_+ \end{bmatrix}$$

and $\bar{\rho} \in \mathbb{R}^{4n+m}$ and $v \in \mathbb{R}^n$ are arbitrary vectors independent of x and $\tilde{x}$, σ_1 and σ_2 are pre-defined scalar and $X_+ = X(\tilde{A}(x)x)$.

Proof: Consider the state tracking error as
$$e(k) = x(k) - x_r(k) \tag{9}$$

It is clear that the output tracking error of the closed-loop system can be written as $e_y(k) = Ce(k)$. Substituting $x(k)$ from (9) in (7) and using from the fact that $e_y(k) = Ce(k)$ we have

$$\begin{aligned} x(k+1) = {}&(A(x(k)) + B(x(k))F(h)C)e(k) \\ &+ (A(x(k)) + B(x(k))G(h)C)x_r(k) \end{aligned} \tag{10}$$

Consider

$$V(e(k)) = e^T(k)X^{-1}(\tilde{x}(k))e(k) \tag{11}$$

as the polynomial Lyapunov function candidate. Therefore

$$\begin{aligned} \Delta V &= V\big(e(k+1)\big) - V\big(e(k)\big) \\ &= e^T(k+1)X^{-1}\big(\tilde{x}(k+1)\big)e(k+1) \\ &\quad - e^T(k)X^{-1}\big(\tilde{x}(k)\big)e(k) \end{aligned} \tag{12}$$

It can be seen that
$$\begin{aligned} e(k+1) &= C\big(x(k+1) - x_r(k+1)\big) \\ &= (A(x) + B(x)F(h)C)\,e(k) \\ &\quad + (A(x) - A_r + B(x)G(h)C)x_r(k) \\ &\quad - B_r r(k) \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned} &\Delta V \\ &= e^T(k+1)X^{-1}\big(\tilde{A}(x)x\big)e(k+1) \\ &\quad - e^T(k)X^{-1}(\tilde{x})e(k) \\ &= \big((A(x) + B(x)F(h)C)e(k) \\ &\quad + (A(x) - A_r + B(x)G(h)C)x_r(k) \\ &\quad - B_r r(k)\big)^T X^{-1}\big(\tilde{A}(x(k))x(k)\big)\big((A(x) \\ &\quad + B(x)F(h)C)e(k) \\ &\quad + (A(x) - A_r + B(x)G(h)C)x_r(k) - B_r r(k)\big) \\ &\quad - e^T(k)X^{-1}(\tilde{x}(k))e(k) \\ &= (e^T(k)(A(x) + B(x)F(h)C)^T \\ &\quad + x_r(k)^T(A(x) - A_r + B(x)G(h)C)^T \\ &\quad - r(k)^T B_r{}^T)X^{-1}\big(\tilde{A}(x(k))x(k)\big)\big((A(x) \\ &\quad + B(x)F(h)C)e(k) \\ &\quad + (A(x) - A_r + B(x)G(h)C)x_r(k) - B_r r(k)\big) \\ &\quad - e^T(k)X^{-1}(\tilde{x}(k))e(k) \end{aligned} \tag{14}$$

Selecting $P_+ = X^{-1}\big(\tilde{A}(x(k))x(k)\big)$ and $P = X^{-1}(\tilde{x}(k))$ the last equation can be written in the matrix form of

$$\begin{aligned} &\Delta V \\ &= [e^T(k) \quad x_r^T(k) \quad r^T] \\ &* \begin{bmatrix} (A(x) + B(x)F(h)C)^T P_+(A(x) + B(x)F(h)C) - P \\ (A(x) - A_r + B(x)G(h)C)^T P_+(A(x) + B(x)F(h)C) \\ -B_r{}^T P_+(A(x) + B(x)F(h)C) \\ (A(x) + B(x)F(h)C)^T P_+(A(x) - A_r + B(x)G(h)C) \\ (A(x) - A_r + B(x)G(h)C)^T P_+(A(x) - A_r + B(x)G(h)C) \\ -B_r{}^T P_+(A(x) - A_r + B(x)G(h)C) \\ -(A(x) + B(x)F(h)C)^T P_+ B_r \\ -(A(x) - A_r + B(x)G(h)C)^T P_+ B_r \\ B_r{}^T P_+ B_r \end{bmatrix} * \begin{bmatrix} e \\ x_r(k) \\ r \end{bmatrix} \end{aligned} \tag{15}$$

Consider H_∞ tracking performance related to the tracking error $x(k) - x_r(k)$ as ([26])

$$\sum_{k=0}^{k_f} \{e^T(k)Qe(k)\} < \sigma^2 \sum_{k=0}^{k_f} w(k)^T w(k) \tag{16}$$

where $w(k)$ is exogenous signal , k_f is the final time of control, Q is a positive definite weighting matrix and σ is a prescribed attenuation level.

By substituting $w(k) = [x_r(k), r(k)]^T$ in (16) and selecting $\sigma = \text{diag}(\sigma_1, \sigma_2)$ we have:

$$\sum_{k=0}^{k_f} (e^T(k)Qe(k) - \sigma_1^2 x_r(k)^T x_r(k) - \sigma_2^2 r(k)^T r(k)) \tag{17}$$
$$< 0$$

Now consider

$$\Delta V + \sum_{k=0}^{k_f} \{e^T(k)Qe(k) - \sigma_1^2 x_r(k)^T x_r(k) \tag{18}$$
$$- \sigma_2^2 r(k)^T r(k)\} < 0$$

Substituting ΔV from (15) in to (18) results

$$[e^T(k) \quad x_r^T(k) \quad r^T(k)] *$$
$$\begin{bmatrix} (A(x) + B(x)F(h)C)^T P_+(A(x) + B(x)F(h)C) - P + Q \\ (A(x) - A_r + B(x)G(h)C)^T P_+(A(x) + B(x)F(h)C) \\ -B_r{}^T P_+(A(x) + B(x)F(h)C) \\ (A(x) + B(x)F(h)C)^T P_+(A(x) - A_r + B(x)G(h)C) \\ (A(x) - A_r + B(x)G(h)C)^T P_+(A(x) - A_r + B(x)G(h)C) - \sigma_1^2 I \\ -B_r{}^T P_+(A(x) - A_r + B(x)G(h)C) \end{bmatrix}$$

$$\begin{bmatrix} -(A(x) + B(x)F(h)C)^T P_+ B_r \\ -(A(x) - A_r + B(x)G(h)C)^T P_+ B_r \\ B_r{}^T P_+ B_r - \sigma_2^2 I \end{bmatrix} * \begin{bmatrix} e(k) \\ x_r(k) \\ r(k) \end{bmatrix} < 0 \qquad (19)$$

The last inequality holds if and only if

$$\begin{bmatrix} (A(x) + B(x)F(h)C)^T P_+ (A(x) + B(x)F(h)C) - P + Q \\ (A(x) - A_r + B(x)G(h)C)^T P_+ (A(x) + B(x)F(h)C) \\ -B_r{}^T P_+ (A(x) + B(x)F(h)C) \\ (A(x) + B(x)F(h)C)^T P_+ (A(x) - A_r + B(x)G(h)C) \\ (A(x) - A_r + B(x)G(h)C)^T P_+ (A(x) - A_r + B(x)G(h)C) - \sigma_1^2 I \\ -B_r{}^T P_+ (A(x) - A_r + B(x)G(h)C) \\ -(A(x) + B(x)F(h)C)^T P_+ B_r \\ -(A(x) - A_r + B(x)G(h)C)^T P_+ B_r \\ B_r{}^T P_+ B_r - \sigma_2^2 I \end{bmatrix} < 0 \qquad (19)$$

It can be seen that the last matrix inequality can be written as

$$[(A(x) + B(x)F(h)C)^T \quad ((A(x) - A_r) + B(x)G(h)C)^T \quad -B_r^T]$$
$$* P_+ * (P_+^{-1} P_+) *$$
$$[(A(x) + B(x)F(h)C) \quad ((A(x) - A_r) + B(x)G(h)C) \quad -B_r]$$
$$- \begin{bmatrix} (P - Q) & 0 & 0 \\ 0 & \sigma_1^2 I & 0 \\ 0 & 0 & \sigma_2^2 I \end{bmatrix} < 0 \qquad (20)$$

By applying Schur complement on (20) it can be concluded that

$$\begin{bmatrix} (P - Q) & & 0 \\ 0 & & \sigma_1^2 I \\ 0 & & 0 \\ P_+\big((A(x) + B(x)F(h)C)\big) & P_+\big((A(x) - A_r) + B(x)G(h)C\big) \\ & 0 & * \\ & 0 & * \\ & \sigma_2^2 I & * \\ & -P_+ B_r & P_+ \end{bmatrix} \succ 0 \qquad (21)$$

Using the congruence transformation with $T = \mathrm{diag}(P^{-1}, I, I, P_+^{-1})$ on the last inequality results

$$\begin{bmatrix} P^{-1}(P - Q)P^{-1} \\ 0 \\ 0 \\ \big((A(x) + B(x)F(h)C)\big)P^{-1} \\ 0 & 0 & * \\ \sigma_1^2 I & 0 & * \\ 0 & \sigma_2^2 I & * \\ \big((A(x) - A_r) + B(x)G(h)C\big) & -B_r & P_+^{-1} \end{bmatrix} \succ 0 \qquad (22)$$

Selecting $F(h)CP^{-1} = M(x)$ results and using Lemma 2.2

$$\begin{bmatrix} X & & X \\ X & & Q^{-1} \\ 0 & & 0 \\ 0 & & 0 \\ A(x)X + B(x)M(x) & & 0 \\ 0 & 0 & * \\ 0 & 0 & * \\ \sigma_1^2 I & 0 & * \\ 0 & \sigma_2^2 I & * \\ \big((A(x) - A_r) + B(x)G(h)C\big) & -B_r & X_+ \end{bmatrix} \succ 0 \qquad (23)$$

Matrix inequality (23) can be solved using SOS programming solvers to find matrices X, M and G. □

Remark 1: In the case where all of the states of the system are presented in the output vector ($C = I$) the controller gain can be calculated from

$$F = MX^{-1} \qquad (24)$$

Remark 2: Consider the case where $C = $

$[I_{q \times q} \quad 0_{(n-q) \times (n-q)}]$. In this case, one can select $X(\tilde{x})$ as $X(\tilde{x}) = \mathrm{diag}(X_{11}, X_{22}(\tilde{x}))$ where $X_{11} \in \mathbb{R}^{q \times q}$ and $X_{22} \in \mathbb{R}^{(n-q) \times (n-q)}$. Therefore the controller gain can be calculated from

$$M = FCX = FC\,\mathrm{diag}(X_{11}, X_{22}(\tilde{x}))$$
$$F = MX_{11}^{-1} \qquad (25)$$

Remark 3: Let $C^T C$ be full rank. Select $F = F' * (C^T C)^{-1} C^T$. It is clear that $F' \in \mathbb{R}^{m \times n}$ and

$$M = FCX = F'X$$
$$F' = MX^{-1} \qquad (26)$$

Remark 4: Let CC^T be full rank. Therefore the controller gain can be calculated from

$$M = FCX$$
$$F = MX^{-1}C^T(CC^T)^{-1} \qquad (27)$$

IV. SIMULATION EXAMPLES

In this section, some examples are presented to illustrate the merits of the proposed approach. The third-party MATLAB toolbox SOSTOOLS [4] is employed to find the feasible solution of *Theorem* 1 numerically. After finding the feasible solution for decision matrices P, M and G the remaining controller gain (F) can be computed using *Remark* 1 to *Remark* 4 .

A. Example1:

Consider the discrete-time nonlinear dynamics of the tunnel diode circuit sampled at T as ([23]):

$$x_1(k + 1) = x_1(k) + T[-0.1x_1(k) - 0.5x_1^3(k) + 50x_2(k)]$$
$$x_2(k + 1) = x_2(k) + T[-x_1(k) - x_2(k) + u(k)] \qquad (28)$$
$$y(k) = Cx(k), \qquad C = I, \quad D = 0$$

The stable reference model for this example is chosen as a linear system with $A_r = \begin{bmatrix} 0.9994 & 0.05 \\ -0.001 & 0.99 \end{bmatrix}$ and $B_r = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}$. In this example we choose $T = 0.001$. Moreover, choosing $X(\tilde{x})$, $M(h)$ and $N(h)$ as constant matrices, $\sigma_1 = 2$, $\sigma_2 = 1$ and $Q = \mathrm{diag}(10^{-3}, 10^{-1})$, the controller gains are obtained as:

$$F(h) = [-2.846 \quad -755.1]$$
$$G(h) = [0.0008 \quad 8.283]$$

Moreover, $P^{-1} = X = \mathrm{diag}(0.6357, 0.2089)$. The controller is employed to control the nonlinear plant subject to the initial condition $x_0 = [0.1 \quad 0.05]$. On the other hand, the initial condition of the referenced model is chosen as $x_{r0} = [0.2 \quad -0.1]$. The system responses to sinusoidal input $r(k) = \sin(5k)$ and step input are shown in Fig.1 and Fig.2 respectively. It can be seen from the figures that the system outputs are able to follow those of the reference model.

B. Example2:

Consider the polynomial system [19]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$$
$$= \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1 x_2^2 + \frac{1}{4}x_2 - x_1^2 x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} \qquad (29)$$
$$+ \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} u$$
$$y_1 = x_1 - x_2$$
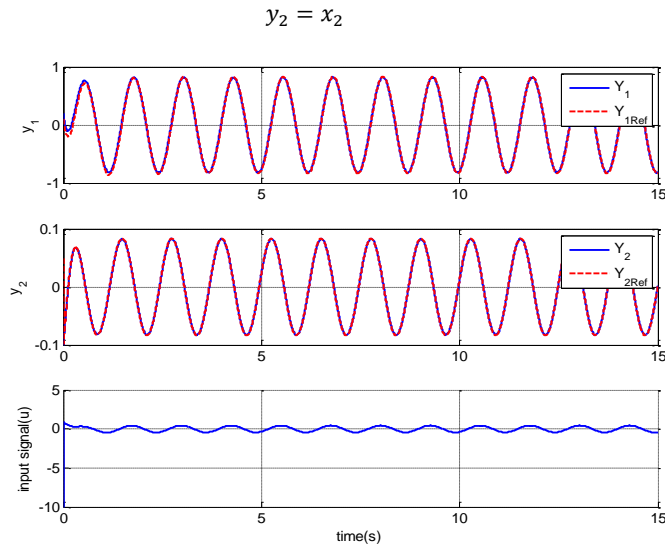
$$y_2 = x_2$$

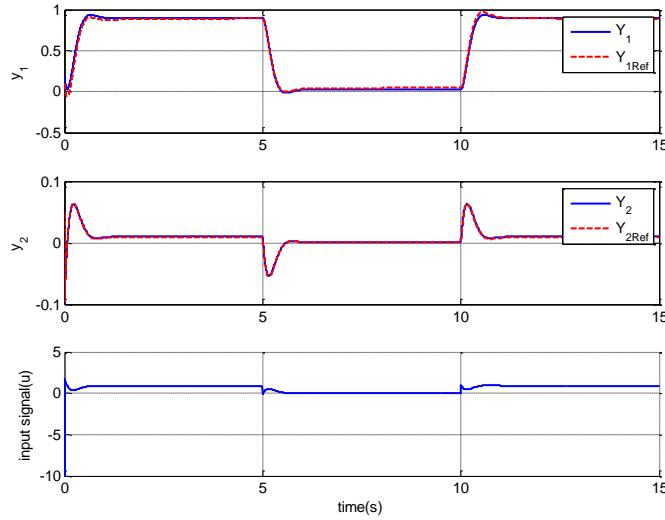Fig.1: Example 1-Sinusoidal reference tracking

Fig.2: Example 1-Step reference tracking

If the system (29) is sampled at T and by Euler's discretization method then the discrete-time dynamic equationscan be derived as:

$$A = \begin{bmatrix} 1 + T[-1 + x_1(k) - \frac{3}{2}x_1^2(k) - \frac{3}{8}x_2^2(k) - x_1(k)x_2(k) \\ 0 \\ T[0.25 - 0.25x_2^2(k)] \\ 1 \end{bmatrix} \quad (30)$$

$$B = \begin{bmatrix} 0 \\ 1.1T \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

In this example, the stable reference model is chosen as $A_r = \begin{bmatrix} 0.75 & 0.0002 \\ 0 & 0.999 \end{bmatrix}$, $B_r = [0 \quad 0.0011]^T$ and sampling period as $T = 0.01$. Given Pas a constant matrix, $M(h)$and $N(h)$as polynomial matrices with degree of 2 where $h = [y_1 \quad y_2]^T$, $\sigma_1 = 3$, $\sigma_2 = 3$ and $Q = \text{diag}(10^{-2}, 10^{-1})$, the controller gains are obtained as:

$$F(h) = [F_1 \quad F_2]$$
$$F_1 = 0.001\ y_1 + 0.002\ y_2 - 5.225 \times 10^{-8}y_1y_2 + 3.672 \times 10^{-7}y_1^2 + 1.022 \times 10^{-10}y_2^2 - 5.861$$
$$F_2 = 0.001\ y_1 + 0.002\ y_2 - 5.196 \times 10^{-8}y_1y_2 + 3.673 \times 10^{-7}y_1^2 + 1.103 \times 10^{-10}y_2^2 - 437.2$$

$$G(h) = [G_1 \quad G_2]$$
$$G_1 = -0.002y_1 - 1.589 \times 10^{-9}y_1\,y_2 - 1.733 \times 10^{-9}y_1^2 + 2.04 \times 10^{-10}y_2^2 - 0.3504$$
$$G_2 = -0.002y_1 + 2.004 \times 10^{-9}y_1y_2 - 1.729 \times 10^{-9}y_1^2 + 1.977 \times 10^{-10}y_2^2 - 0.2259$$

Moreover, $P^{-1} = X = \text{diag}(0.3266, 5.88)$. The system responses to sinusoidal input $r(k) = 0.5\sin(5k)$ and step input are shown in Fig.3 and Fig.4. It can be seen from the figures that the system outputs have tracked those of the reference model. In this simulation the initial condition of the plant and the reference model have selected as $x_0 = [0.2 \quad 0.01]$ and $x_{r0} = [-0.2 \quad 0]$, respectively.
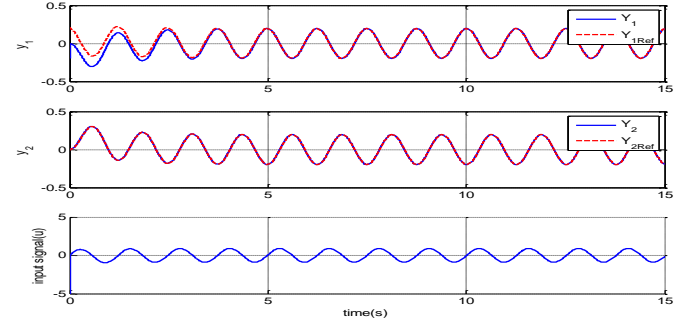

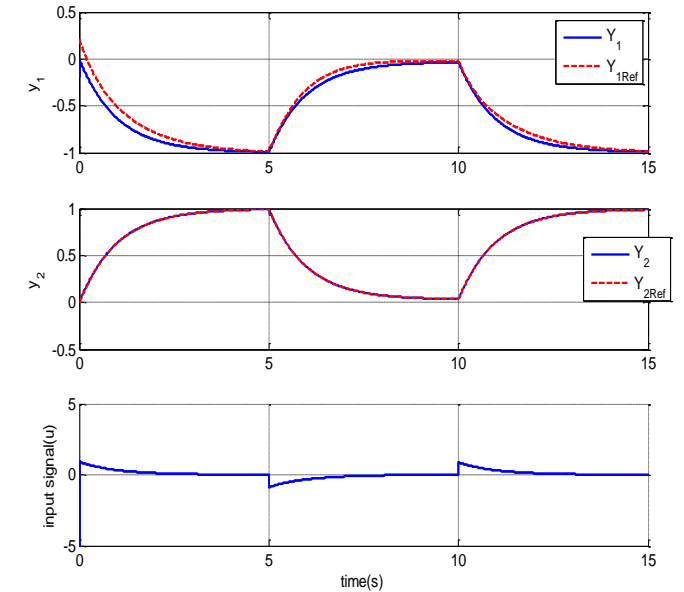Fig.3: Example 2 - Sinusoidal reference tracking

Fig.4:Example 2- Step reference tracking

C. Example3:

Consider the chaotic system (Henon mapping) [27]
$$x_1(k + 1) = -x_1^2(k) + 0.3x_2(k) + u(k)$$
$$x_2(k + 1) = x_1(k) \quad (31)$$
$$y = x_1(k)$$

In this example, the stable reference model is chosen as $A_r = \begin{bmatrix} 0.6 & 0.3 \\ 1 & 0 \end{bmatrix}$, $B_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Choosing $X(\tilde{x})$ as a constant matrix, $M(h)$ and $N(h)$ as polynomial matrices with degree of 2 where $h = y$, $\sigma_1 = 4$, $\sigma_2 = 4$ and $Q = \text{diag}(1,1)$, the controller gains are calculated as:

$$F(h) = -8.992 \times 10^{-12}y^2 + y + 0.003552$$
$$G(h) = -2.2.6 \times 10^{-12}y^2 + y + 0.6983$$

Moreover we have $P^{-1} = X = \mathrm{diag}(0.13, 0.24)$. Setting the initial condition $x_{r0} = [-0.5 \quad -0.1]$ and $x_0 = [0.5 \quad 0]$ for reference model and plant respectively, the system responses to sinusoidal input $r(k) = 0.5\sin(5k)$ and step input are shown in Fig.5 and Fig.6. It can be seen from the figures that the system output has tracked that of the stable reference model.
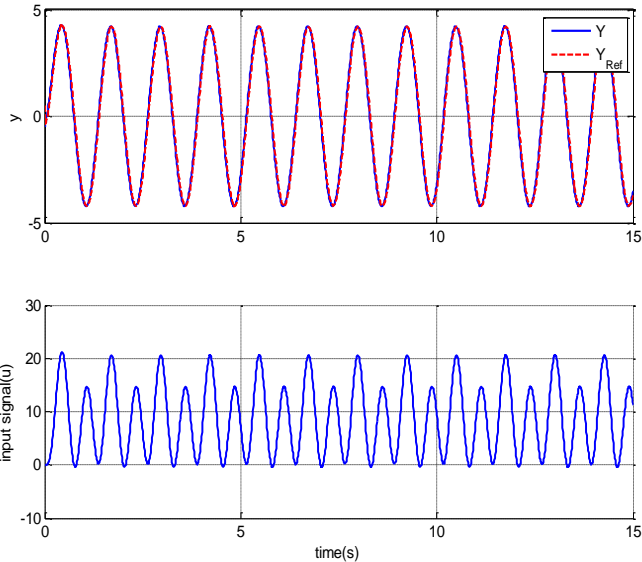

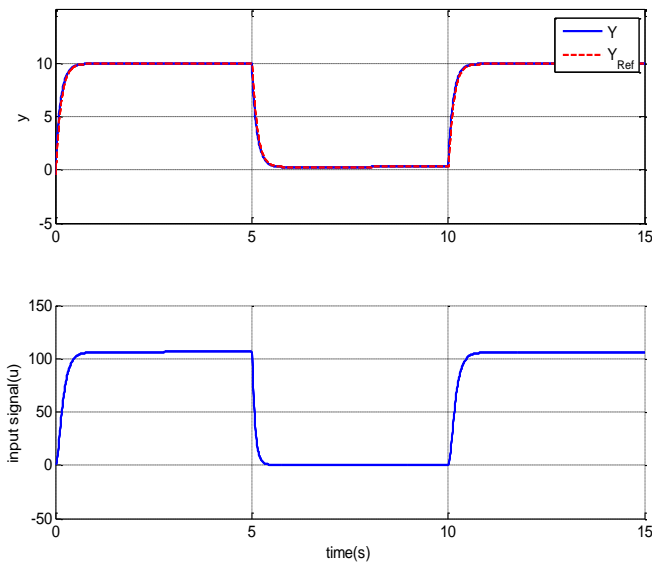Fig.5: Example 3-Sinusoidal reference tracking

Fig.6: Example 3-Step reference tracking

I.V. CONCLUSION

In this paper we studied tracking control synthesis problem for nonlinear polynomial discrete-time systems. The proposed controller drove the plant such that its state vector followed those of a stable reference model. Sufficient conditions for the existence of controller presented in terms of constraints of a feasibility SOS program that could be solved using numerical solvers such as SOSTOOLS. We discussed the proposed approach for the two case of state and output feedbacks. Finally, the merits of the proposed approach showed using the simulation of several examples.

REFERENCES

[1] P.A. Parrilo, "Structured Semidefinite Programs and Semi algebraic Geometry Methods in Robustness and Optimization", Ph.D. dissertation, California Institute of Technology, Pasadena, 2000.

[2] D. Henrion, A. Garulli, Positive Polynomials in Control, Springer-Verlag Berlin Heidelberg, 2005.

[3] Z. Jarvis-Wloszek, R. Feeley, T. Weehong, S. Kunpeng, A. Packard, "Some controls applications of sum of squares programming," in IEEE Conference on Decision and Control, 2003, pp. 4676-4681.

[4] S. Prajna, A. Papachristodoulou, P.A. Parrilo, Introducing "SOSTOOLS: a general purpose sum of squares programming solver," in: IEEE Conference Decision and Control, 2002, pp. 741-746.

[5] J. Lofberg, "YALMIP : a toolbox for modeling and optimization in MATLAB," in IEEE International Symposium on Computer Aided Control Systems Design, 2004, pp. 284-289.

[6] D. Henrion, J.-B. Lasserre, "GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi," *ACM Transactions on Mathematical Software*, vol.29, no.2, pp. 165-194, 2003.

[7] S. Prajna, A. Papachristodoulou, W. Fen, "Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach," in *5'th Asia Control Conference*, 2004, pp.157-165.

[8] J. D. Zhao, L. Wang, "Robust static output feedback design for polynomial nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol.20 no.14, pp.1637–1654, 2010

[9] Y. Zhi, G. Zhao, a.J. Yu, "Robust static output feedback for a class of nonlinear uncertain systems," in *International Conference on Computational and Information Sciences*, 2011, pp. 1163-1166.

[10] A. Papachristodoulou, S. Prajna, "A tutorial on sum of squares techniques for systems analysis," in *American Control Conference*, 2005, pp. 2686-2700.

[11] G. Chesi, "LMI techniques for optimization over polynomials in control," *IEEE Transaction on Automatic Control*, vol.55 no.11, pp. 2500-2510, 2010.

[12] X. Jun, X. Lihua, W. Youyi, "Simultaneous Stabilization and Robust Control of Polynomial Nonlinear Systems Using SOS Techniques," *IEEE Transactions on Automatic Control*, vol. 54, no. 8, pp. 1892-1897, 2009.

[13] Z. Qian, W. Fen, "Nonlinear output feedback H_∞ control for polynomial nonlinear systems," in *American Control Conference*, 2008, pp. 1196-1201.

[14] U. Topcu, A. Packard, P. Seiler, "Local stability analysis using simulations and sum-of-squares programming," *Automatica*, vol. 44, no. 10, 2669-2675, 2008.

[15] Z. Dan, W. Jianliang, "An improved nonlinear H_∞ synthesis for parameter-dependent polynomial nonlinear systems using SOS programming," in *American Control Conference*, 2009, pp. 796-801.

[16] D. Zhao, J. Wang, "An improved H_∞ synthesis for parameter-dependent polynomial nonlinear systems using SOS programming," in *American Control Conference*, 2009, pp. 796-801.

[17] H. Wen-Chao, S. Hong-Fei, Z. Jian-Ping, "Robust Control Synthesis of Polynomial Nonlinear Systems Using Sum of Squares Technique," *Acta Automatica Sinica*, vol. 39, no. 6, 799-805, 2013.

[18] G. Franzè, D. Famularo, "A robust fault detection filter for polynomial nonlinear systems via sum-of-squares decompositions," *Systems & Control Letters*, vol.61, no.8, pp. 839–848, 2012

[19] M. Krug, S. Saat, N.S. Kiong, "Robust H_∞ static output feedback controller design for parameter dependent polynomial systems: An iterative sums of squares approach," *Journal of the Franklin Institute*, vol. 350, no. 2, pp. 318-330, 2012.

[20] J. Xu, L. Xie, Y. Wang, "Synthesis of Discrete-time Nonlinear Systems: A SOS Approach," in *American Control Conference*, 2007, pp. 4829-4834.

[21] H. J. Ma, G.H. Yang, "Fault tolerant H_∞ control for a class of nonlinear discrete-time systems: Using sum of squares optimization," in *American Control Conference*, 2008, pp. 1588-1593.

[22] S. Saat, D. Huang, S.K. Nguang, A.H. Hamidon, "Nonlinear state feedback control for a class of polynomial nonlinear discrete-time

systems with norm-bounded uncertainties: An integrator approach," *Journal of the Franklin Institute*, vol.350, no.7, 1739-1752, 2013.

[23] S. Saat, S.K. Nguang, A.M. Darsono, N. Azman, "Nonlinear H_∞ feedback control with integrator for polynomial discrete-time systems," *Journal of the Franklin Institute*, vol. 315, pp. 4023-4038, 2014.

[24] S. Saat, D. Huang, S.K. Nguang, "Robust State Feedback Control Of Uncertain Polynomial Discrete-Time Systems: An Integral Action Approach," *International Journal of Innovative Computing, Information and Control*, vol. 9, no.3, 1233-1244, 2013.

[25] S. Boyd, L. Ghaoui, E. Feron, V. Balakrishnan, *Linear matrix inequalities in system and control theory*, SIAM, 1994.

[26] B. S. Chen, C. H. Lee, Y.C. Chang, "H_∞ tracking design of uncertain nonlinear SISO systems: Adaptive fuzzy approach," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 1, 32–43, 1996.

[27] C.-L. Chen, G. Feng, D. Sun, X.-P. Guan, "H_∞ Output Feedback Control of Discrete-Time Fuzzy Systems With Application to Chaos Control," *IEEE Transactions on Fuzzy Systems*, vol. 13, no.4, pp. 531-543, 2005.