Fuzzy Sliding-Mode State-Feedback Control of Non-Minimum Phase Quadruple Tank System

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Abstract

This paper considers control of a laboratory Quadruple Tank System (QTS) in its non-minimum phase mode. This system is a well-known laboratory process suitable to illustrate the concepts of multivariable control methods. The objective of this paper was to design a controller based on combination of the sliding-mode and the state-feedback control methods using fuzzy logic. The proposed method takes advantage of the fast transient response of the sliding-mode controller and the zero steady-state error of the state-feedback controller. In other words, the fuzzy system uses the SMC when the QTS is in the transient mode and utilizes the SFC when it is near the steady-state mode. Hence, the advantages of both controllers have been used simultaneously. The switching between these two controllers is continuous and smooth based on a few simple fuzzy rules. Stability analysis of the proposed method is presented based on the Lyapunov stability direct method. Experimental results confirmed effectiveness of the proposed method as compared with the stand-alone controllers, especially when there are uncertainties in the system parameters.

Keywords: non-minimum phase system, quadruple tank system, sliding-mode control, state-feedback, control, hybrid fuzzy systems

1. Introduction

The Quadruple Tank System (QTS) is a well-known multivariable laboratory process system that has been widely used by many researchers as a benchmark problem for multivariable control methods, especially when the system operates in its non-minimum phase mode, where the linearized dynamics of the process exhibits a multivariable zero on the right-hand side of the *s*-plane. This situation is achieved by adjusting the valves position of the QTS [1, 2]. This feature has attracted researchers to control this process in its minimum as well as non-minimum phase modes. Better understanding of the QTS mechanism and designing different controllers by developing mathematical model of this nonlinear system has been proposed in literature. Ucak and Oke have presented e-Support Vector Regression (SVR) method to model a quadruple tank system [3]. They have suggested the use of SVR in intelligent modeling of nonlinear systems and tuning the controller parameters based on the system model. Yuntao *et al.* have proposed two multivariable model-based control schemes for QTS. A MIMO model predictive and network-based H_∞ multivariable tracking controllers have been implemented on the experimental QTS in [4].

Traditional modeling techniques are rather complex and time consuming and incorporate the entire dynamics of the process. However, soft computing techniques can approximates the process using input-output data sets. Malar has employed soft computing techniques (i.e., Neural Networks (NNs), fuzzy logic, and their combinations) for the QTS modeling [5]. Fuzzy logic has also been used for controlling of the QTS. Malar and Thyagarajan have proposed decentralized fuzzy pre-compensated PI controller for the QTS in both the minimum and non-minimum phase modes [6]. They have employed relative-gain array analysis [7] for decentralized control of this process and have shown that in the non-minimum phase mode, the input-output pairing should be reversed in order to create smoother outputs without oscillations, which would increase the actuators life time. Pourmohammad *et al.* have established an adaptive fuzzy multivariable controller design based on the Lyapunov scheme, where the closed-loop system is monitored and the parameters of the controller are adapted in order to minimize the tracking error [8].

The Model Predictive Control (MPC) method is an advanced control strategy that uses a model to predict the future behavior of the system. Deepa *et al.* have developed a discrete-time MPC for the QTS with and without a dead time in the process [9]. The control vector is optimized and the results are compared with the decentralized PI controller. Alvardo *et al.* have proposed a robust tube-based MPC for tracking piecewise constant references based on the quadruple tank process [10]. They have shown that the controller ensures

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the feasibility for an admissible set point and convergences to a desired steady state. Pan and Wang have used two NNs in the MPC and tested their method on the QTS [11]. In this line of work, Sivakumaran proposed a nonlinear MPC using recurrent NN as a predictor and implemented that on the QTS [12]. Mercangoz *et al.* have proposed a distributed MPC and applied it on an experimental QTS in the non-minimum phase mode [13]. They have partitioned the system into self-sufficient estimation and control nodes. The results of their experiments have shown that their method performs better than a completely decentralized set of controllers. In [14], authors have proposed distributed multi-parametric MPC for the QTS that exhibits some robustness against uncertainties in the system parameters.

Shneiderman and Palmor have extended the QTS to include multivariable dead times that may introduce infinite, finite or no non-minimum phase zeros [15]. They have shown that existence of non-minimum phase zeros depends on a particular combination of multivariable dead times.

Biswas *et al.* have developed a sliding-mode controller (SMC) for the QTS in its non-minimum phase mode based on the feedback linearization method [16]. Although the proposed method is robust, the presence of discontinuous functions in the controller creates chattering, which is undesirable for system actuators. To reduce this effect, they have considered the well-known boundary layer around the sliding surface that creates steady-state errors. Gareli *et al.* have proposed a collective SMC for the QTS in its minimum phase mode [17].

In [18], the quadruple tank process has been exploited to study different types of nonlinear observers and three approaches namely Extended Kalman Filter (EKF), High-Gain Observer (HGO), and High-Gain EKF (HG-EKF) have been analyzed.

Gaaloul *et al.* have presented an output feedback controller for the QTS to estimate the water levels into two bottom tanks using the information available from the measured levels [19, 20]. The same control technique has been employed in [21] and [22], where the level control for the QTS with variable nonlinear zero dynamics has been considered and the interactions and effects of the uncertainties have been discussed.

An inherent property of the multivariable systems is the interaction between their deferent inputs and outputs. Gareli *et al.* have presented a partial decoupling method for the MIMO systems and have implemented it to the non-minimum phase of the QTS. They have shown that the switching is carried out at very high frequencies [23].

In 2002, Åström *et al.* have proposed a PI controller with interacting loops for an experimental QTS [24]. In this line of work, in 2004, Numsomran *et al.* have proposed PI controller for the QTS by using decentralized methods for the non-minimum phase and minimum phase modes [25].

In this paper, the objective is to combine the SMC and the State-Feedback Controller (SFC) using fuzzy logic for the non-minimum phase mode of the QTS. Moreover, uncertainties in the system are also considered. The SMC is employed because of its fast transient response and robustness against uncertainties in the system, while the SFC is used because of its zero steady-state error. Hence, the fuzzy system uses the SMC when the QTS is in the transient mode and utilizes the SFC when it is near the steady-state mode. In this way, the advantages of both controllers are used simultaneously.

This paper is organized as follows. Section 2 describes the nonlinear QTS. In Section 3, modeling and parameter estimation of the QTS will be given. Sections 4 describes the design of the SMC, the SFC, and the hybrid controller based on the fuzzy logic. Stability analysis of the proposed controller is presented in Section 5. Section 6 shows the experimental result. Section 7 concludes the paper.

2. Rrocessdescription

The QTS has two pumps that are used to discharge water from reservoir into four overhead tanks [1]. The schematic diagram of this system is shown in Figure 1. One of the interesting characteristics of this system is placing one of its multivariable zeros on either half of the *s*-plane by changing the position of two valves. The manipulated variables of the QTS are the voltages applied to the pumps. The controlled variables are the water levels in two lower tanks (i.e., tanks 1 and 2). The output of each pump is divided into two tanks; one in the lower part and the other in the upper part, diagonally opposite. In other words, the outflow of pump 1 splits between tanks 1 and 4 and the outflow of pump 2 splits between tanks 2 and 3 (Figure 1). The split ratio is determined by the valves positions. The quadruple tank process has two transmission zeros. The position of one of these zeros depends on the split fraction g_1 and g_2 in valves 1 and 2, respectively. The minimum and non-minimum phase modes can be achieved as

$$\begin{array}{ll} \text{minimum phase mode:} & 1 < (g_1 + g_2) < 2 \\ \text{non-minimum phase mode:} & 0 < (g_1 + g_2) < 1 \end{array} \tag{1}$$

The non-minimum phase mode of this system (i.e., when there exist right-half-plane zeros) imposes serious limitations on the performance of the controller and makes the control problem more challenging. In this paper, only the non-minimum phase mode of the QTS is considered.

3. Modeling and Parameter stimation

The governing dynamical equations of the QTS can be represented as follows [1, 2]:

$$\begin{aligned} \frac{dh_1}{dt} &= \frac{-a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{g_1 k_1}{A_1} v_1 \\ \frac{dh_2}{dt} &= \frac{-a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{g_2 k_2}{A_2} v_2 \\ \frac{dh_3}{dt} &= \frac{-a_3}{A_3}\sqrt{2gh_3} + \frac{(1-g_2)k_2}{A_3} v_2 \\ \frac{dh_4}{dt} &= \frac{-a_4}{A_4}\sqrt{2gh_4} + \frac{(1-g_1)k_1}{A_4} v_1 \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \end{aligned} \tag{2}$$

where h_i, a_i and A_i $(i = 1, \mathbf{K}, 4)$ are the water level, the cross section of the outlet hole and the cross section of the corresponding tank, respectively, g_1 and g_2 are the split coefficients of valves 1 and 2, respectively, v_1 and v_2 are the voltages applied to the pumps, respectively, k_1 and k_2 are constants relating the control voltages with water flow from the pumps and g is the gravitational constant.

For designing the State-Feedback Controller (SFC), all parameter values of the system are needed because this method depends on the model of the system. For this reason, parameters k_1, k_2, g_1 and g_2 in (2) need to be estimated. The parameter values are estimated using the ***idnlgrey*** function in MATLAB software. This function finds a Grey-Box model that describes the system behavior as a set of nonlinear differential equations with unknown parameters. The structure of this function is

$$m = \text{idnlgrey}('\,filename',\ order,\ parameters,\ initial\ \ states,\ Ts)$$

Appendix I gives more details of this parameter estimation. For parameter estimation, input-output data are collected from the laboratory system by applying step inputs to the pumps (e.g., $u_1 = 3$ V and $u_2 = 2.8$ V). Model validation is shown in Figure 2. As this figure shows, the ***idnlgrey*** function has estimated the unknown parameters with approximately 84% accuracy. The estimated parameters are presented in Table 1, where $0 < g_1 + g_2 \approx 0.76 < 1$, which shows that the system operates in its non-minimum phase mode.

4. Controller Design

4.1. Design of SMC

The standard normal form for a 2×2 MIMO system can be represented as [16]

$$\begin{aligned} \dot{x}_1^1 &= x_2^1 \\ \dot{x}_2^1 &= \ddot{x}_1^1 = f_1^1(x) + g_1^1(x)\, u_1 + g_2^1(x)\, u_2 \\ \dot{x}_1^2 &= x_2^2 \\ \dot{x}_2^2 &= \ddot{x}_1^2 = f_1^2(x) + g_1^2(x)\, u_1 + g_2^2(x)\, u_2 \end{aligned} \tag{3}$$

where $\mathbf{x} = [x_1^1\ x_2^1\ x_1^2\ x_2^2]^{\mathrm{T}}$ is the state vector and $\mathbf{y} = [x_1^1\ x_1^2]^{\mathrm{T}}$ is the output vector.

Equation (2) is not in the form of (3) and hence, should be transformed to the standard normal MIMO form. In non-minimum phase mode, the manipulated variables u_1 and u_2 have negligible effects on the water levels of the two bottom tanks since their dynamics are mainly controlled by the water flow from their respective upper tanks. Hence, the flow ratio from their direct pump can be ignored. Thus, for determining the relative degree of the system for designing the SMC, inputs u_1 and u_2 must appear in the controlled variables h_1 and h_2 [26]. Therefore, by taking derivatives of $\dot h_1$ and $\dot h_2$ in (2), it gives

$$\begin{aligned}
\dot y_1 &= \dot x_1^1 = x_2^1 \\
\dot x_2^1 &= \ddot h_1 = \left(\frac{a_1^2 g}{A_1^2} - \frac{a_1 a_3 g\sqrt{h_3}}{A_1^2\sqrt{h_1}} - \frac{a_3^2 g}{A_1 A_3} \right) + \\
&\quad \left(-\frac{a_1 g \gamma_1 k_1}{A_1^2\sqrt{2gh_1}} \right) u_1 + \left(\frac{a_3 g(1-\gamma_2)k_2}{A_1 A_3\sqrt{2gh_3}} \right) u_2 \\
\dot y_1 &= \dot x_1^2 = x_2^2 \\
\dot x_2^2 &= \ddot h_2 = \left(\frac{a_2^2 g}{A_2^2} - \frac{a_2 a_4 g\sqrt{h_4}}{A_2^2\sqrt{h_2}} - \frac{a_2^4 g}{A_2 A_4} \right) + \\
&\quad \left(\frac{a_4 g(1-\gamma_1)k_1}{A_2 A_4\sqrt{2gh_4}} \right) u_1 + \left(-\frac{a_2 g \gamma_2 k_2}{A_2^2\sqrt{2gh_2}} \right) u_2
\end{aligned} \tag{4}$$

According to (4), the relative degree of QTS in non-minimum phase mode is equal to two. Based on the sliding surface equation

$$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} e \tag{5}$$

the sliding surfaces for the MIMO QTS can be written as

$$\begin{aligned} s_1 &= \lambda e_1 + \dot e_1 \\ s_2 &= \lambda e_2 + \dot e_2 \end{aligned} \tag{6}$$

where λ is a positive constant and $e = x - x_d$ is the tracking error. Sufficient condition for reaching the error trajectories on the sliding surfaces and staying on them is that the manipulated variables u_1 and u_2 are designed such that the following sliding condition is satisfied:

$$\frac{1}{2}\frac{d}{dt}\left(s_1^2 + s_2^2\right) \le -\eta_1 |s_1| - \eta_2 |s_2| \quad \forall t > 0 \tag{7}$$

where η_1 and η_2 are small positive constants.

By considering uncertainties for f_1^1 and f_1^2 in (3) as $\hat f_1^1$ and $\hat f_1^2$, respectively, the upper bound of uncertainties can be defined as

$$|\hat f_1^i - f_1^i| \le F_i, \quad i = 1, 2 \tag{8}$$

The uncertainties on the input vector can be considered as

$$\mathbf{G}(\mathbf{x}) - \hat{\mathbf{G}}(\mathbf{x}) = \Delta\hat{\mathbf{G}}(\mathbf{x}) \tag{9}$$

where $\mathbf{G}(x) = \begin{bmatrix} g_1^1(x) & g_2^1(x) \\ g_1^2(x) & g_2^2(x) \end{bmatrix}$.

It can be shown that the sliding control law can be derived as [27]

$$\mathbf{u}_{\mathrm{SMC}} = \begin{bmatrix} -\frac{a_1 g \gamma_1 k_1}{A_1^2\sqrt{2gh_1}} & \frac{a_3 g(1-\gamma_2)k_2}{A_1 A_3\sqrt{2gh_3}} \\ \frac{a_4 g(1-\gamma_1)k_1}{A_2 A_4\sqrt{2gh_4}} & -\frac{a_2 g \gamma_2 k_2}{A_2^2\sqrt{2gh_2}} \end{bmatrix}^{-1} \begin{bmatrix} \ddot h_{1d} - \left[\left(\frac{a_1^2 g}{A_1^2} - \frac{a_1 a_3 g\sqrt{h_3}}{A_1^2\sqrt{h_1}} - \frac{a_3^2 g}{A_1 A_3} \right) + \lambda_1 \dot e_1 \right] \\ - k_{s1}\,\mathrm{sgn}(s_1) \\ \ddot h_{2d} - \left[\left(\frac{a_2^2 g}{A_2^2} - \frac{a_2 a_4 g\sqrt{h_4}}{A_2^2\sqrt{h_2}} - \frac{a_4^2 g}{A_2 A_4} \right) + \lambda_2 \dot e_2 \right] \\ - k_{s2}\,\mathrm{sgn}(s_2) \end{bmatrix} \tag{10}$$

where $\mathbf{k}_s = [k_{s1}\ k_{s2}]^{\mathrm{T}}$ satisfies the reaching condition (7) with the sign function defined as

$$\mathrm{sgn}(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases} \tag{11}$$

4.2. Design of SFC

Using the Taylor series expansion, (2) can be linearized around its equilibrium point and the state-space realization of the QTS can be written as [1]

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{12}$$

where

$$\mathbf{A}=\begin{bmatrix}-\frac{1}{T_1} & 0 & \frac{1}{T_3} & 0\\ 0 & -\frac{1}{T_2} & 0 & \frac{1}{T_4}\\ 0 & 0 & -\frac{1}{T_3} & 0\\ 0 & 0 & 0 & -\frac{1}{T_4}\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}\frac{g_1k_p}{A_1} & 0\\ 0 & \frac{g_2k_2}{A_2}\\ 0 & \frac{(1-g_2)k_2}{A_2}\\ \frac{(1-g_1)k_1}{A_1} & 0\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$$

$$T_i=\frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}\quad (i=1,\mathbf{K},4)$$

in which $h_i^0 \ (i=1,\mathbf{K},4)$ are the equilibrium points of QTS. The objective of implementing the state-feedback controller is to minimize the following performance index [27]:

$$J=\int_0^\infty(\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x}+\mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u})dt$$

where **Q** and **R** are constant and positive-definite matrices. The optimal control law is

$$\mathbf{u}_{\mathrm{SFC}}=-\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x} \tag{13}$$

where **P** is a symmetric positive-definite matrix that satisfies the following algebraic Riccati equation:

$$-\mathbf{PA}-\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{PBR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}-\mathbf{Q}=0.$$

Hence, the SFC for the QTS is

$$\mathbf{u}_{\mathrm{SFC}}=-\mathbf{K}^{\mathrm{T}}\mathbf{x} \tag{14}$$

where the gain matrix **K** is equal to

$$\mathbf{K}=-\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} \tag{15}$$

4.3. Design of hybrid controller

In this section, a combination of the SMC and the SFC with the aid of a fuzzy system will be presented. It is well known that the SMC has a fast transient response and is robust against uncertainties in the system. However, when the system trajectories are near the sliding surfaces the chattering phenomenon occurs. By introducing a boundary layer around the operating point, as it is common among researchers, the chattering can be avoided. However, there will be steady-state errors. In order to eliminate the chattering and at the same time obtaining zero steady-state response, the SFC will be used when the QTS operates near the sliding surfaces. The switching between these two controllers is performed using a fuzzy system.

The fuzzy IF-THEN rules for the hybrid controller are defined as

$$\begin{aligned}&\text{Rule 1: IF } |e| \text{ is H, THEN } \mathbf{u}=\mathbf{u}_{\mathrm{SMC}}\\ &\text{Rule 2: IF } |e| \text{ is L, THEN } \mathbf{u}=\mathbf{u}_{\mathrm{SFC}}\end{aligned} \tag{16}$$

where $|e|$ is the absolute value of the tracking error and H and L are fuzzy variables designating high and low values, respectively. The membership functions of these fuzzy variables are shown in Figure 3. Since there are two manipulated variables in the QTS, two fuzzy systems are needed. In these fuzzy systems, $|e|$ is defined as the corresponding fuzzy input variable. As discussed in Section 4, when the system is in the non-minimum phase mode, the water level of the lower tanks is mainly controlled by the outflow of their respective upper tanks. Thus, in the fuzzy system, $|e_1|$ is the input to the fuzzy controller for generating u_2 while $|e_2|$ is the input to the fuzzy system for generating u_1. When the states of the system are far from the operating point, the first rule in (16) is triggered and hence, the SMC is applied to the system. On the other hand, when the states of the system are near the sliding surface (i.e., near the operating point) the second rule is activated and the SFC is applied to the system. Finally, when the states of the system are neither far from the operating point and nor near it, a combination of the SMC and the SFC is applied to the system. Since a fuzzy system is used for the hybrid controller, the transition between the SMC and the SFC is continuous and smooth. By using the weighted-sum defuzzification method, the inputs to the pumps $[u_1 \;\; u_2]$ can be obtained as

$$u_i(t) = \frac{m_H\left(|e_i|\right)u_{SMC(i)} + m_L\left(|e_i|\right)u_{SFC(i)}}{m_H\left(|e_i|\right) + m_L\left(|e_i|\right)} \quad \left(i=1,2\right). \tag{17}$$

The block diagram of the proposed controller is shown in Figure 4.

5. STability analysis

The stability analysis is based on breaking down the control problem into two fuzzy subsystems. Since the stability of every fuzzy subsystem is checked individually, the complexity of analysis is simplified. However, the condition that every fuzzy subsystem yields a stable closed-loop system does not directly imply that the entire fuzzy system, composed of several subsystems, yields a stable closed-loop system as well. Sufficient conditions that make this implication valid are stated in the following theorem:

Theorem

Consider a hybrid fuzzy control system as given in (16) and (17). If

(1) there exists a positive-definite, continuously differentiable, and radially unbounded scalar function $V = \mathbf{x}^{\mathrm{T}}\mathbf{P}\,\mathbf{x}$, where $\mathbf{P} \in R^{n\times n}$ is a constant and positive-definite matrix,

(2) every fuzzy subsystems gives a negative-definite $\dot{V}$ in the active region of the corresponding fuzzy rule, and

(3) the weighted-sum defuzzification method is employed, which for any input, the output u_c of the fuzzy logic controller is between u_p and u_q such that $u_p \le u_c \le u_q$, where u_p and u_q are the lower and upper bound of u_c, respectively,

then, according to the direct Lyapunov theory, the equilibrium point at the origin is globally asymptotically stable.

Proof: See [28] and [29].

Therefore, to guarantee the system stability, we need to find a suitable common Lyapunov function V and ensure that every fuzzy subsystem yields a negative-definite $\dot{V}$ in the active region of the corresponding fuzzy rule. The active region of the corresponding fuzzy rule is defined as the region $X_r \subset X$ such that the membership function $m_i(x_0)$ is not zero for all $x_0 \in X_r$.

5.1. SMC subsystem

Consider the Lyapunov function

$$V = \frac{1}{2}(s_1^2 + s_2^2) \tag{18}$$

Using (3) and (6), the time-derivative of (18) becomes

$$\dot{V} = \left[s_1 \;\; s_2\right]\begin{bmatrix}\dot{s}_1\\ \dot{s}_2\end{bmatrix}$$

Defining $E_1 := \dot{e}_1 + l\, e_1 - \ddot{y}_{1d}$ and $E_2 := \dot{e}_2 + l\, e_2 - \ddot{y}_{2d}$, it gives

$$\dot{V} = \left[s_1 \;\; s_2\right]\begin{bmatrix}\ddot{y}_1 + E_1\\ \ddot{y}_2 + E_2\end{bmatrix} = \left[s_1 \;\; s_2\right]\left(\begin{bmatrix}f_2\\ f_4\end{bmatrix} + \mathbf{G}\mathbf{u} + \begin{bmatrix}E_1\\ E_2\end{bmatrix}\right) \tag{19}$$

Consider the following sliding control law:

$$\mathbf{u}_{\mathrm{SMC}} = \hat{\mathbf{G}}^{-1}\begin{bmatrix}-\hat{f}_2 - E_1 - k_{s1}\,\mathrm{sgn}(s_1)\\ -\hat{f}_4 - E_2 - k_{s2}\,\mathrm{sgn}(s_2)\end{bmatrix}. \tag{20}$$

Applying (20) to (19) yields

$$\dot{V} = \left[s_1 \;\; s_2\right]\left(\begin{bmatrix}f_2\\ f_4\end{bmatrix} + \mathbf{G}\hat{\mathbf{G}}^{-1}\begin{bmatrix}-\hat{f}_2 - E_1 - k_{s1}\,\mathrm{sgn}(s_1)\\ -\hat{f}_4 - E_2 - k_{s2}\,\mathrm{sgn}(s_2)\end{bmatrix} + \begin{bmatrix}E_1\\ E_2\end{bmatrix}\right) \tag{21}$$

Using (9) and substituting $\mathbf{G}\hat{\mathbf{G}}^{-1} = \mathbf{I} + \mathbf{\Delta}$ in (21) gives

$$
\begin{aligned}
\dot{V} &= [s_1\ \ s_2]\left(\begin{bmatrix} f_2 \\ f_4 \end{bmatrix} + \begin{bmatrix} 1+\Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1+\Delta_{22} \end{bmatrix}\right. \\
&\qquad\left.\begin{bmatrix} -\hat{f}_2 - E_1 - k_{s1}\operatorname{sgn}(s_1) \\ -\hat{f}_4 - E_2 - k_{s2}\operatorname{sgn}(s_2) \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\right) \\
&= [s_1\ \ s_2]\begin{pmatrix} (f_2-\hat{f}_2) - \Delta_{11}(\hat{f}_2 - E_1) - \Delta_{12}(\hat{f}_4 - E_2) \\ -k_{s1}(1+\Delta_{11})\operatorname{sgn}(s_1) - k_{s2}\Delta_{21}\operatorname{sgn}(s_2) \\ (f_4-\hat{f}_4) - \Delta_{21}(\hat{f}_2 - E_1) - \Delta_{22}(\hat{f}_4 - E_2) \\ -k_{s1}\Delta_{21}\operatorname{sgn}(s_1) - k_{s2}(1+\Delta_{22})\operatorname{sgn}(s_2) \end{pmatrix}
\end{aligned} \tag{22}
$$

Applying the defined boundary in (8) to (22) yields

$$
\begin{aligned}
\dot{V} \le &\Big(F_2 + g_{11}\left|\hat{f}_2 - E_1\right| + g_{12}\left|\hat{f}_4 - E_2\right| \\
&\qquad - (1-g_{11})k_{s1} + k_{s2}g_{12}\Big)|s_1| \\
&+ \Big(F_4 + g_{21}\left|\hat{f}_2 - E_1\right| + g_{22}\left|\hat{f}_4 - E_3\right| \\
&\qquad + g_{11}k_{s1} - (1-g_{22})k_{s2}\Big)|s_2|
\end{aligned} \tag{23}
$$

If the following conditions are satisfied

$$
\begin{aligned}
(1-g_{11})k_{s1} &\ge F_2 + g_{11}\left|\hat{f}_2 - E_1\right| + g_{12}\left|\hat{f}_4 - E_2\right| \\
&\qquad + g_{12}k_{s2} + h_1 \\
(1-g_{22})k_{s2} &\ge F_4 + g_{21}\left|\hat{f}_2 - E_1\right| + g_{22}\left|\hat{f}_4 - E_2\right| \\
&\qquad + g_{11}k_{s1} + h_2
\end{aligned} \tag{24}
$$

then, the sliding mode reaching condition (7) is verified

$$
\dot{V} \le -h_1|s_1| - h_2|s_2| \tag{25}
$$

Hence, the stability of the SMC subsystem is guaranteed.

5.2. SFC subsystem

For the SFC subsystem, the same Lyapunov function as in (18) is used

$$
V = \frac{1}{2}\mathbf{s}^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{s} \tag{26}
$$

where $\mathbf{s} = [s_1\ \ s_2]^{\mathrm{T}}$. In order to guarantee the stability of the SFC subsystem, let us define new state variable as $\dot{e}_1$ and $\dot{e}_2$. This is because the derivative of these two states exist in $\dot{V}$. The augmented open-loop plant can be represented as

$$
\frac{d}{dt}\begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_3 \\ e_4 \\ e_2 \\ \dot{e}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ m_1 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & m_6 & m_5 & 0 \end{bmatrix}}_{\mathbf{A_p}}\begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_3 \\ e_4 \\ e_2 \\ \dot{e}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ b_1 & b_2 \\ 0 & b_3 \\ b_4 & 0 \\ 0 & 0 \\ b_5 & b_6 \end{bmatrix}}_{\mathbf{B_p}}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{27}
$$

The parameters in (27) are given in Appendix II.
Assuming the linear system as $\dot{\mathbf{x}} = \mathbf{A}_p\mathbf{x} + \mathbf{B}_p\mathbf{u}$, and by selecting $\mathbf{e} = \mathbf{x} - \mathbf{x}_d$, the error dynamics can be written as

$$
\dot{\mathbf{e}} = \mathbf{A}_p\mathbf{e} + \mathbf{B}_p\mathbf{u} + \mathbf{A}_p\mathbf{x}_d \tag{28}
$$

It can be shown that if $\mathbf{A}\mathbf{x}_d = 0$ and if the state variables of the new system in (28) converge to zero, then the state variable of the system will converge to $\mathbf{x}_d$. It is straightforward to shown that the following state-feedback control law can derive the state variables in (28) to zero:

$$
\mathbf{u}_{\mathrm{SFC}} = \left(\underbrace{-\frac{1}{R}\mathbf{B}^{\mathrm{T}}\mathbf{P}}_{\mathbf{k}_p}\right)\mathbf{e} \tag{29}
$$

Since $(\mathbf{A}_P, \mathbf{B}_P)$ is controllable, it is possible to move the eigenvalues of $\mathbf{A}_P - \mathbf{B}_P\mathbf{K}_p$ to the desired locations. Hence, the stable closed-loop dynamics can be presented as

$$
\dot{\mathbf{e}} = \underbrace{(\mathbf{A}_P - \mathbf{B}_P\mathbf{K}_p)}_{\mathbf{A}_{\mathrm{cl}}}\mathbf{e} \tag{30}
$$

Now, the sliding surfaces (6) is defined as

$$s_1 = \begin{bmatrix} l & 1 & 0_{1\times4} \end{bmatrix} \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_3 \\ e_4 \\ e_2 \\ \dot{e}_2 \end{bmatrix}, s_2 = \begin{bmatrix} 0_{1\times4} & l & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_3 \\ e_4 \\ e_2 \\ \dot{e}_2 \end{bmatrix} \quad (31)$$

By defining

$$\mathbf{\Lambda} := \begin{bmatrix} l & 1 & 0_{1\times4} \\ 0_{1\times4} & l & 1 \end{bmatrix} \quad (32)$$

and substituting (31) into (26), it yields

$$V = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{\Lambda}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{e} \quad (33)$$

where $\mathbf{P} := \mathbf{\Lambda}^{\mathrm{T}}\mathbf{\Lambda}$ is a symmetric positive definite matrix. The time derivative of (33) is

$$\begin{aligned} \dot{V} &= \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\mathbf{P}\mathbf{e} + \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{A}_{\mathrm{cl}}^{\mathrm{T}}\mathbf{P}\mathbf{e} + \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{cl}}\mathbf{e} \\ &= \frac{1}{2}\mathbf{e}^{\mathrm{T}}(\mathbf{A}_{\mathrm{cl}}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}_{\mathrm{cl}})\mathbf{e} \end{aligned} \quad (34)$$

The stability of $\mathbf{A}_{\mathrm{cl}}$ ensures the existence of the symmetric positive definite solution $\mathbf{P}$ of the following algebraic Riccati equation:

$$\mathbf{P}\mathbf{A}_{\mathrm{cl}} + \mathbf{A}_{\mathrm{cl}}^{\mathrm{T}}\mathbf{P} = -\mathbf{Q} \quad (35)$$

where $\mathbf{Q}$ is a symmetric positive definite matrix. Hence,

$$\dot{V} = -\frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{Q}\mathbf{e} \le 0 \quad (36)$$

Therefore, the state feedback subsystem is also stable.

Then, according to Theorem 1, the whole closed-loop system, consisting of the SMC and SFC sub-controllers, is asymptotically stable.

6. Experimintal Results

The Experimental QTS (Figure 5) has 4 translucent tanks each with a pressure sensor to measure the water level. Two pumps are submersible in water reservoir and allow variable flow control. The height of each tank is 30 cm. The parameter values of the QTS in the non-minimum phase mode are given in Table 2.

The SMC is applied to the system with the switching parameters selected as $k_{s1} = k_{s2} = 1$ and $l = 10$. In addition, the gain matrix $\mathbf{K}$ in the SFC part is determined as

$$\mathbf{K} = \begin{bmatrix} 0.2422 & 0.1954 & 0.0522 & 0.1541 \\ 0.1932 & 0.1409 & 0.1675 & -0.0144 \end{bmatrix}$$

It should be mentioned that the same parameters are used in experiments for the stand-alone SMC and SFC controllers as well as for the hybrid fuzzy SMC-SFC controller.

As depicted in Figures 6 and 7, the hybrid controller has better performance as compared to the SMC and SFC alone. The SMC has undesirable overshoots and chattering in the control signals, which can damage the pumps in a short time. Moreover, the SFC responses are not as fast as the SMC and have larger rise times. On the other hand, the hybrid SMC-SFC controller has better response as compared to both controllers. Table 3 summarizes the performance of different controllers. Figure 8 shows the contribution of the SMC and the SFC in the hybrid fuzzy controller during the operation of the system.

Next, performance of the controllers is investigated against changes in the system parameters. For this reason, the cross section of the outlet hose of pump 1 is changed by about 30%. As Figure 9 shows, the effect of this uncertainty in the steady state is much larger on the SFC than on the hybrid fuzzy controller. This is mainly because the SFC depends on the model of the system and considerable changes in the system parameters can deteriorate performance of the controller. Figures 10 and 11 show that the hybrid controller continuously switches between the SFC and the SMC in order to overcome this large uncertainty.

6. Conclusions

In this paper, a combination of the SFC and the SMC using fuzzy logic was presented for better performance of the nonlinear and non-minimum phase quadruple tank system. The proposed controller has the advantages of both SMC and SFC. In other words, the fast transient response and robustness of the SMC and zero steady-state error of the SFC can be obtained. It was shown by laboratory experiments that the proposed method offers fast transient response as well as insignificant steady-state errors without any chattering in the control signals. Moreover, the combined controller could cope with system uncertainties better than the SFC. The stability of the closed-loop system was shown using the Lyapunov direct method.

8. Reference

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Table 1. Estimated parameters of QTS

g_1	g_2	k_1	k_2
0.42	0.34	27.43	19.55

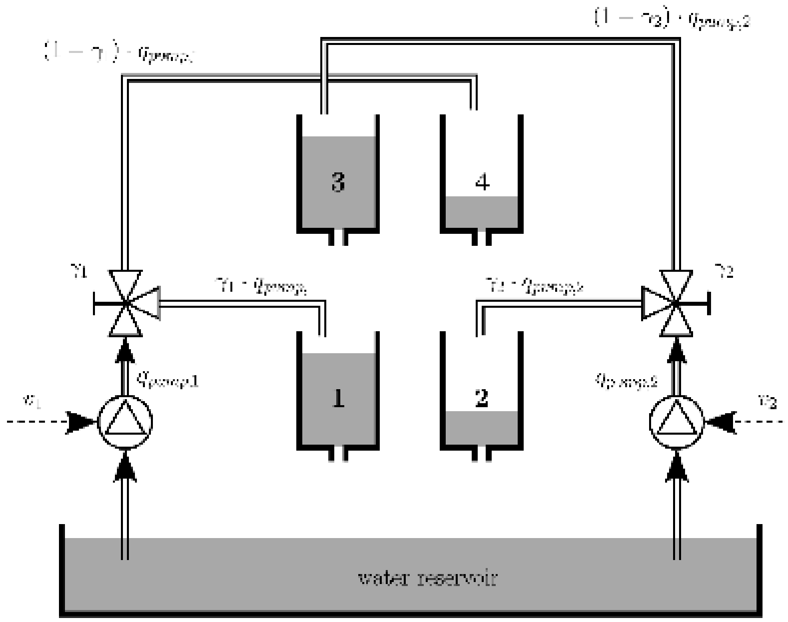


Figure1. Schematic diagram of QTS.

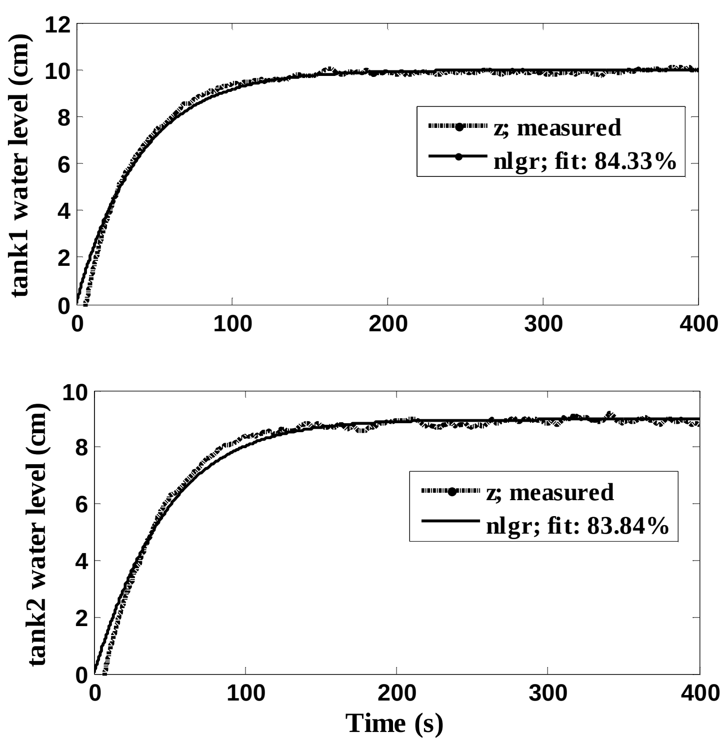


Figure 2. Model validation.

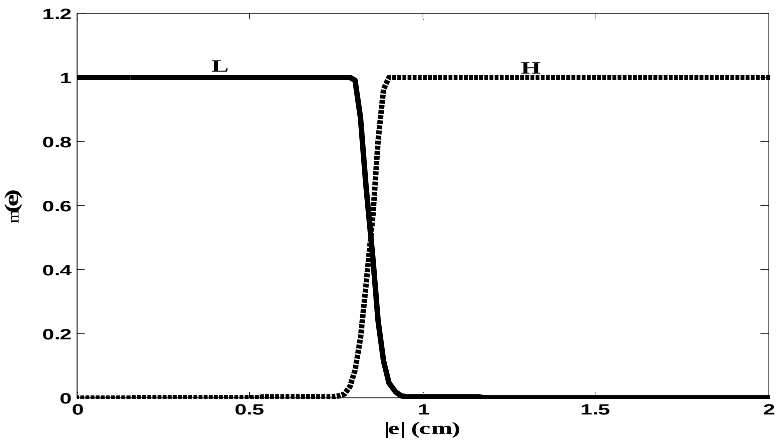


Figure 3. Input membership functions

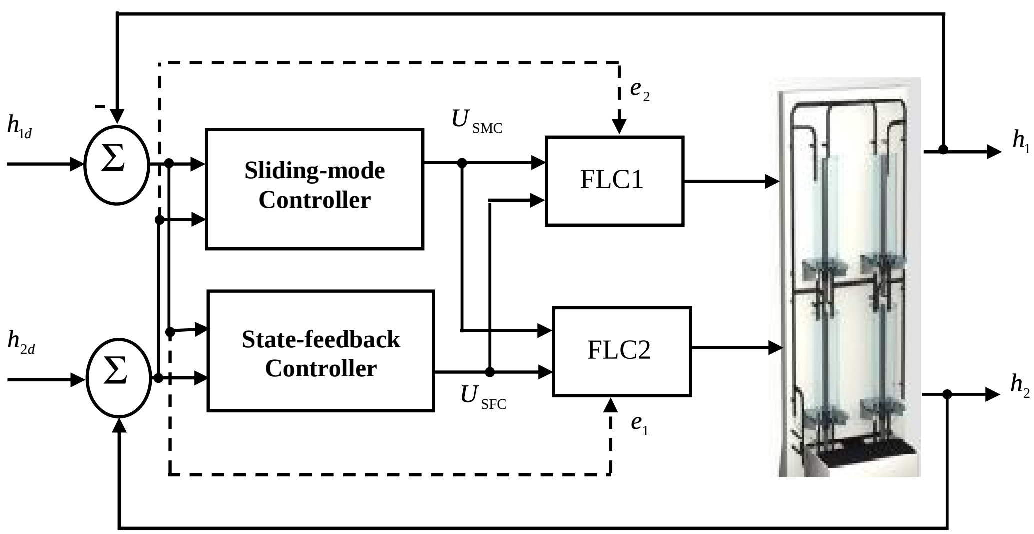
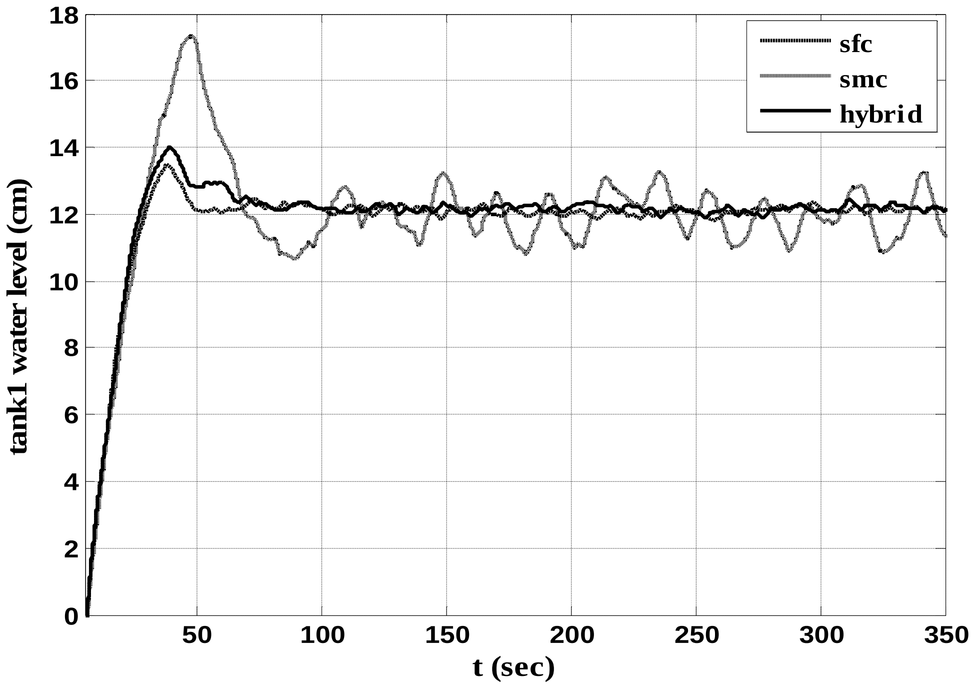


Figure 4. Block diagram of the fuzzy SMC-SFC controller.

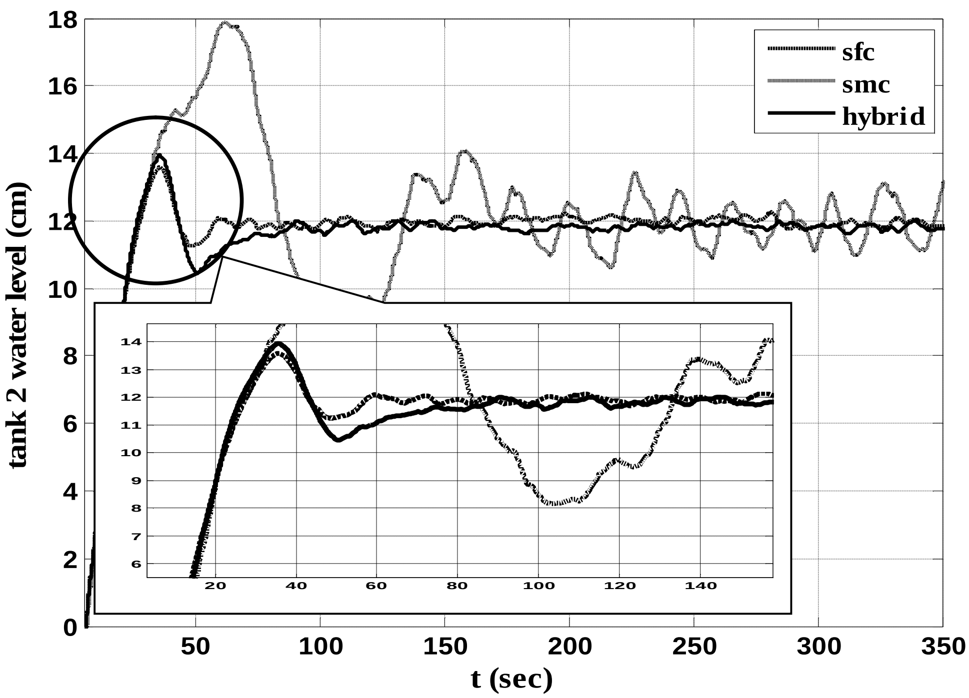
Table 2. Parameter values of QTS

Parameter	Value
Cross section of tanks $A_i \quad (i=1,\dots,4)$	$138.9\ (\mathrm{cm}^2)$
Cross section of outlet hose $a_i \quad (i=1,\dots,4)$	$0.50265\ (\mathrm{cm}^2)$
g	$981\ (\mathrm{cm/s}^2)$

Figure 5. Experimental QTS.

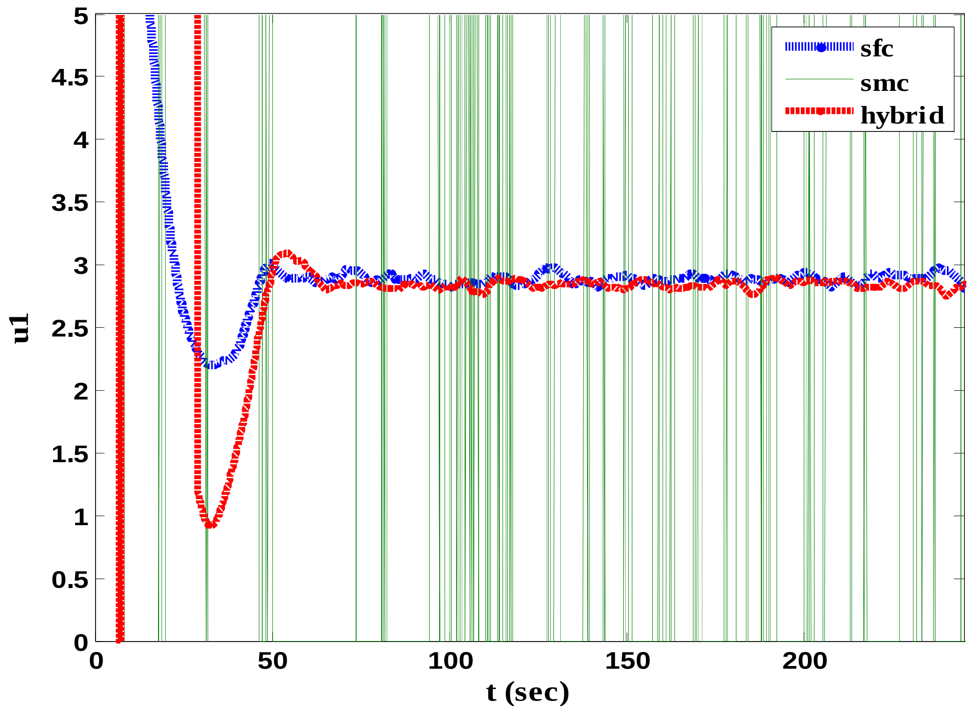


(a)

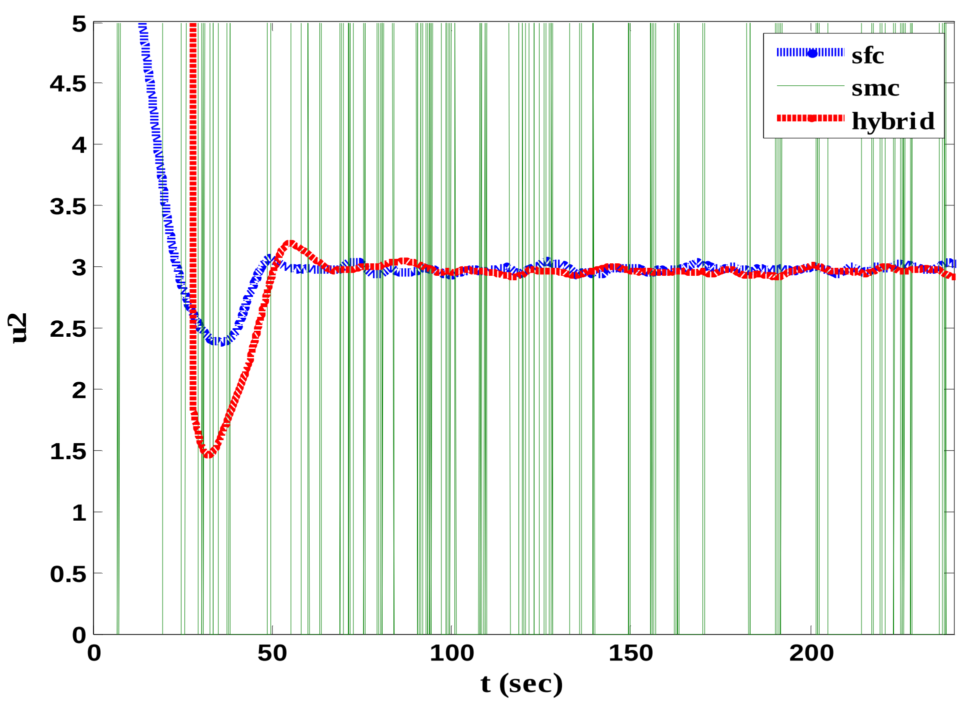


(b)

Figure 6. Performance of three controllers. The desired value for both tanks is 12 cm, (a) Water level in tank 1 and (b) Water level in tank 2.



(a)



(b)

Figure 7. Inputs u_1 and u_2 of different controllers

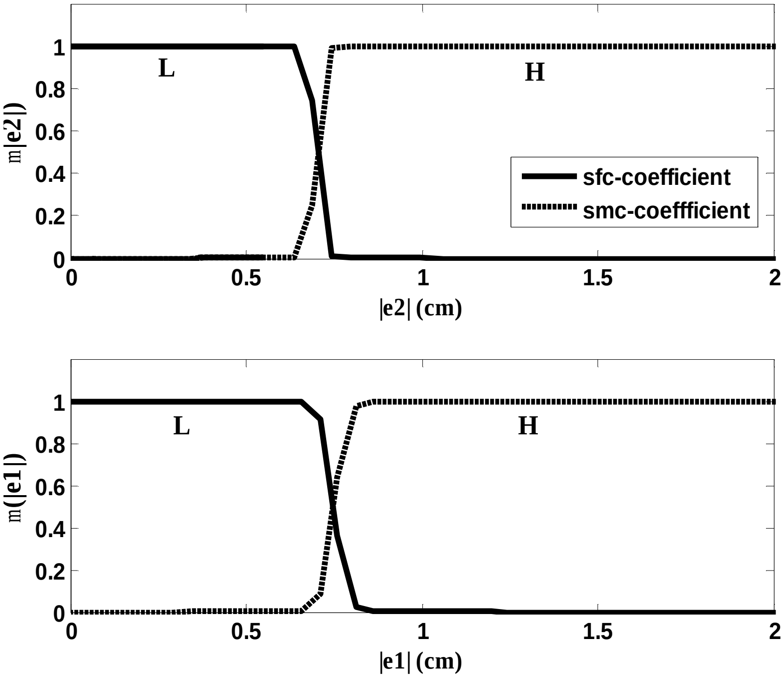


Figure 8.

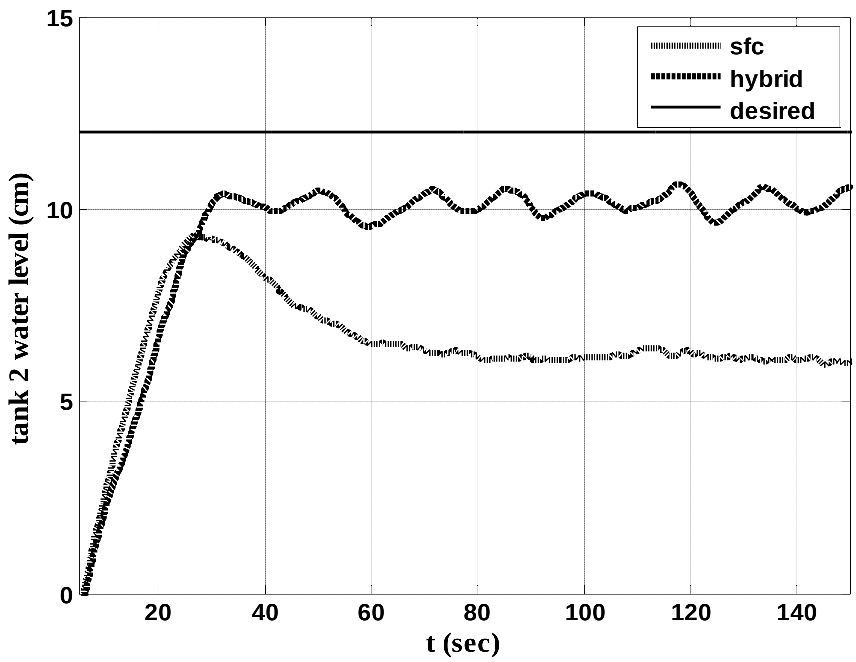


Figure 9. Comparison of SFC and combined controller in presence of system uncertainty.

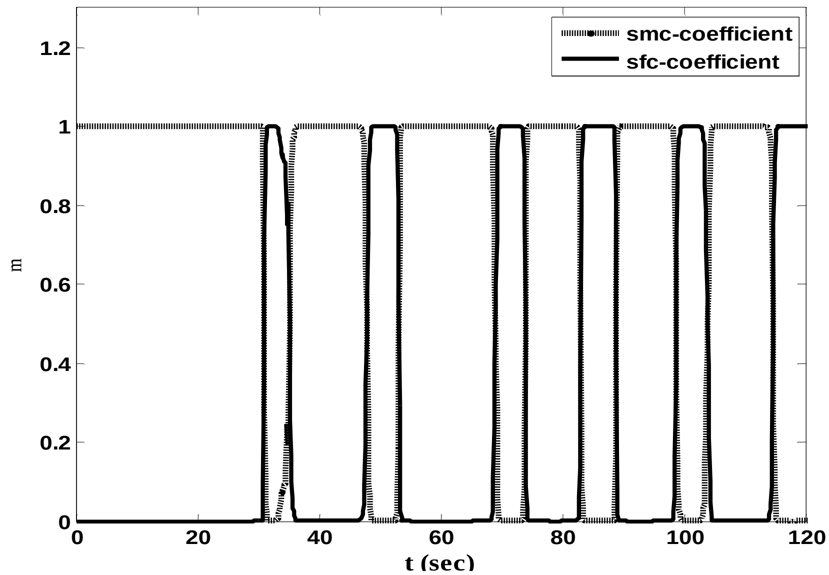


Figure 10. Contribution of SFC and SMC to the hybrid control law for the case of Figure 9.

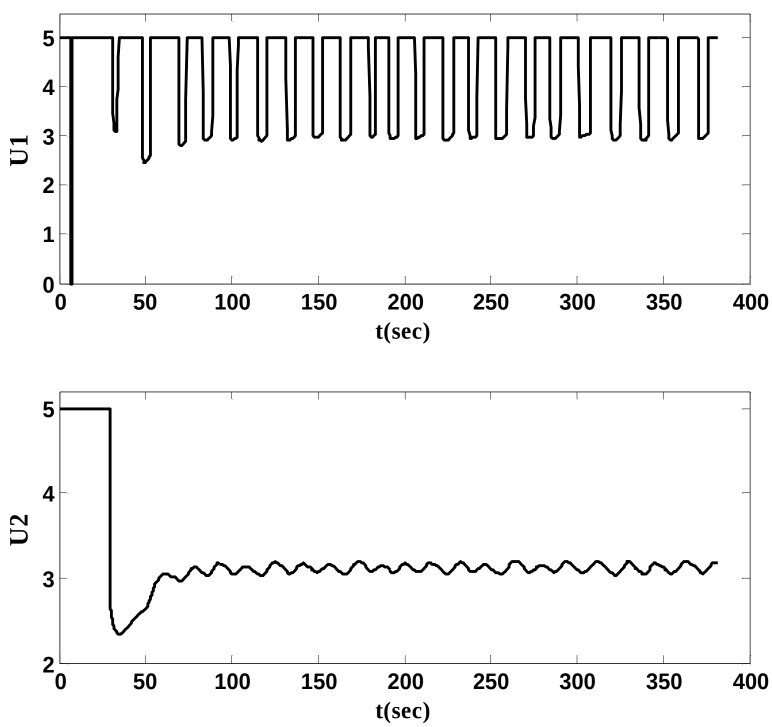


Figure 11. Control inputs for the case of Figure. 9.

Table 3. Quantitative comparison of controllers for tank 1

Controller	Overshoot	Rise time (s)	Settling time (s)	Least mean square error
SMC	44.16%	66	-	0.4275
SFC	10%	80	120	0.0339
Hybrid Controller	13.33%	65	90	0.0331