

# A Comparative Study on $H_\infty$ Resilient Observer Design; Lipschitz and One-sided Lipschitz Fractional Order Systems' State Estimation

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## I. Introduction

**Abstract**— This paper proposes the states estimation of one-sided Lipschitz fractional order systems. In this regard, using an LMI based continuous frequency distribution results in  $H_\infty$  nonlinear resilient observer design in the presence of an exogenous disturbance input and observer's gain perturbation. Since one-sided Lipschitz class of nonlinear systems encompasses a wide range of nonlinearities including the Lipschitz class, It is shown that the proposed observer design based on the one-sided Lipschitz systems has simpler LMI and it can tolerate a wide range of changes in gains perturbation and exogenous disturbances compared to recent research findings for the counterpart Lipschitz systems. Finally, a financial fractional order system with Lipschitz nonlinearity is presented as an example, which can illustrate the effectiveness of the proposed one-sided Lipschitz observer design and compare the feasible region of input disturbance and observer's gain for both Lipschitz and one-sided Lipschitz observer design.

**Index** Fractional order nonlinear systems; Linear matrix inequalities (LMIs); One-sided Lipschitz systems; Robust observer.

Due to the Importance of state estimation in practical and industrial tasks, there are various studies on the design of estimators. In addition, recent engineering calculations have extended to fractional calculus and this has made many scholars focus on fractional order observers' design. On the other hand, while linear systems are not practically accountable, many research studies have been proposed for the state estimation of nonlinear fractional order systems over the last decade [1-5]. A class of nonlinear fractional order observer design has concentrated on Lipschitz systems as a wide range of nonlinear systems [1,2,6,7].

Although working on Lipschitz fractional order observers is still open [8,9], a main drawback in the existing results for Lipschitz nonlinear systems is that they have failed to provide a solution for large

Lipschitz constants and also the reported LMIs cannot be feasible [1]. One-sided Lipschitz nonlinear systems can encompass a more general class of nonlinear systems including Lipschitz nonlinearities. Besides, one-sided Lipschitz constants are significantly smaller than their counterpart Lipschitz constants [1].

Authors in [7] introduced a non-fragile observer design for fractional order one-sided Lipschitz nonlinear systems and the extension of this work is presented in [1] to introduce the full order and reduced order observer for one-sided Lipschitz systems. In continue of the previous researches, in reference [1] the uncertainty of one-sided Lipschitz model is considered in the design of both full order and reduced order observer. One of the newest researches on the fractional order one-sided Lipschitz system is [1], in which Mittag Leffler stability is obtained using Caputo fractional derivative. Nevertheless, these designs can become unstable in the presence of exogenous disturbance input.

To the best of author knowledge, nearly all of the previous researches on fractional order one-sided Lipschitz model have considered both quadratic inner boundedness and one-sided Lipschitz condition for model nonlinearity while [1] has shown this class of nonlinearity is smaller than one-sided Lipschitz.

Motivated by the above discussions, in this paper we consider a fractional order one-sided Lipschitz observer, containing bounded perturbations on the observer gain with exogenous disturbance input. Using a continuous frequency distribution, the stability conditions are derived based on an indirect approach.

An LMI-based  $H_\infty$  optimal observer for state estimation of the one-sided Lipschitz system is proposed, which is robust against perturbations in the gain matrix aside from the presence of exogenous disturbances. Unlike previous researches, this paper suggests a nonlinear observer design for fractional order one-sided Lipschitz system without quadratic inner boundedness and it is robust against an exogenous disturbance input

Financial systems are complicated nonlinear systems that are influenced by external disturbances

due to the factors such as interest rate, the price of goods, investment demand, and stock. On the other hand, some state variables in a financial system are not measureable. Thus, economic adaption and prediction are needed to estimate all the state variables in the presence of uncertainty and disturbances. This system is chosen as a practical model to investigate the proposed design.

The rest of this paper is organized as follows: Section 2 provides the basic concepts. In section 3, the design procedure for nonlinear fractional order observer is discussed, while section 4 presents prior Lipschitz fractional order  $H_\infty$  resilient observer design mathematically and has a theoretical comparison with the main results of this paper in section 3. A financial system is presented as an example in section 5 and finally, the conclusion remarks are given.

## II. Preliminaries and problem statement

In this section, diffusive representation that provides the theoretical basis for a time approximation of fractional order integral is given. Subsequently, the problem statement and some useful Lemmas for our main result are presented.

**Definition.1** [1]: The nonlinear fractional differential equation is considered as:

$${}_a D_t^q X = f(X) \tag{1}$$

due to the continuous frequency distributed model of the fractional integrator, it can be expressed as:

$$\begin{cases} \frac{\partial z(\omega, t)}{\partial t} = -\omega z(\omega, t) + f(X(t)) \\ X(t) = \int_0^\infty \mu(\omega) z(\omega, t) d\omega \end{cases} \tag{2}$$

where  $\mu(\omega)$  is considered as follow:

$$\mu(\omega) = \frac{\sin(q\pi)}{\pi} \omega^{-q} \tag{3}$$

**Definition.2** [1]: The design of the robust proportional observer consists in finding a matrix  $L$  such as the estimation error,  $\tilde{X}(t)$ , satisfies the following  $H_\infty$  performances:

$$\lim_{t \rightarrow \infty} \tilde{X}(t) = 0 \quad \text{for } W(t) = 0$$

$$\|\tilde{X}(t)\|_2^2 \leq \eta \|W(t)\|_2^2 \quad \text{for } \begin{cases} W(t) \neq 0 \\ \text{and} \\ \tilde{X}(0) = 0 \end{cases} \tag{4}$$

Minimizing  $\eta$  causes the smaller state estimation's error.

**Lemma 1 (Schur complement)** [1]: The Linear Matrix Inequality (LMI):

$$\begin{bmatrix} Q(X) & S(X) \\ S^T(X) & R(X) \end{bmatrix} < 0 \tag{5}$$

where  $Q(X) = Q^T(X)$ ,  $R(X) = R^T(X)$ , and  $S(X)$  affinely depend on  $X$ , is equivalent to:

$$\begin{cases} R(X) < 0 \\ Q(X) - S(X)R^{-1}(X)S^T(X) < 0 \end{cases} \tag{6}$$

**Lemma 2:** [18] Let  $X, Y$  be real vectors of the same dimension. Then, for any scalar  $\varepsilon > 0$ , the following inequality holds:

$$X^T Y \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y \tag{7}$$

In the continue, an uncertain fractional order system is considered as:

$$\begin{aligned} D^q X &= AX + BU + \phi(X, U) + W \\ Y &= CX \end{aligned} \tag{8}$$

where  $D^q$  is  $q$  th-order fractional derivative,  $q \in (0, 1]$ ,  $X \in \mathbb{R}^n$ ,  $U \in \mathbb{R}^q$ , and  $Y \in \mathbb{R}^m$  are the state, input, and output, respectively.  $W \in \mathbb{R}^n$  is the disturbance input,  $C \in \mathbb{R}^{m \times n}$  is a constant matrix and  $\phi: [\mathbb{R}^n \times \mathbb{R}^q] \rightarrow \mathbb{R}^n$  is a nonlinear function. The nonlinear function  $\phi(X, U)$  is said to be a Lipschitz function of  $X$  with Lipschitz constants  $\gamma > 0$ , if:

$$\|\phi(X_1, U) - \phi(X_2, U)\| \leq \gamma \|X_1 - X_2\| \quad \forall X_1, X_2 \in \mathbb{R}^n \tag{9}$$

And  $\phi(X, u)$  is said to be a one-sided Lipschitz if there exists a constant  $\rho \in \mathbb{R}$  such that: [1]

$$\begin{aligned} &\langle \phi(X_1, U) - \phi(X_2, U), X_1 - X_2 \rangle \\ &\leq \rho \|X_1 - X_2\|^2, \quad \forall X_1, X_2 \in \mathbb{R}^n \end{aligned} \tag{10}$$

One-sided Lipschitz condition provides a less conservative condition than the counterpart classical Lipschitz [7]. Note that the Lipschitz constant must be positive, while one-sided Lipschitz constant may have



$$A_{\Delta} = ((A - LC - \Delta(t)C)\tilde{X} + \phi(X, U) - \phi(\hat{X}, U) + W)$$

Applying Eq. (16) causes (20) to be simplified as follows:

$$\begin{aligned} \frac{dV(t)}{dt} &= \tilde{X}^T P A_{\Delta} + A_{\Delta}^T P \tilde{X} \\ &\quad - 2 \int_0^{\infty} \omega \mu(\omega) z^T(\omega, t) P z(\omega, t) d\omega \end{aligned} \quad (21)$$

According to the Lyapunov theory, stability condition for system (14) is  $\frac{dV(t)}{dt} < 0$ , i.e. if:

$$\tilde{X}^T P A_{\Delta} + A_{\Delta}^T P \tilde{X} < 0 \quad (22)$$

or equivalently:

$$\begin{aligned} &\tilde{X}^T P(A - LC - \Delta(t)C)\tilde{X} + \tilde{X}^T P W \\ &+ \tilde{X}^T (A - LC - \Delta(t)C)^T P \tilde{X} + W^T P \tilde{X} \\ &+ (\phi(X, U) - \phi(\hat{X}, U))^T P \tilde{X} \\ &+ \tilde{X}^T P(\phi(X, U) - \phi(\hat{X}, U)) < 0 \end{aligned} \quad (23)$$

According to our knowledge  $\tilde{X} \in \mathbb{R}^n$ ,  $P \in \mathbb{R}^{n \times n}$  and  $\phi: [\mathbb{R}^n \ \mathbb{R}^n] \rightarrow \mathbb{R}^n$ , which results in

$$\tilde{X}^T P(\phi(X, U) - \phi(\hat{X}, U)) = ((\phi(X, U) - \phi(\hat{X}, U))^T P \tilde{X})^T.$$

So inequality (23) can be presented as:

$$\begin{aligned} &\tilde{X}^T P(A - LC - \Delta(t)C)\tilde{X} \\ &+ \tilde{X}^T (A - LC - \Delta(t)C)^T P \tilde{X} \\ &+ 2\tilde{X}^T P(\phi(X, U) - \phi(\hat{X}, U)) \\ &+ \tilde{X}^T P W + W^T P \tilde{X} < 0 \end{aligned} \quad (24)$$

Making use of inequality (10), inequality (24) can be simplified to (25):

$$\begin{aligned} &\tilde{X}^T P(A - LC - \Delta(t)C)\tilde{X} + \tilde{X}^T P W + W^T P \tilde{X} \\ &+ \tilde{X}^T (A - LC - \Delta(t)C)^T P \tilde{X} + 2\rho P \|\tilde{X}\|^2 < 0 \end{aligned} \quad (25)$$

Inequality (25) can be rewritten as below:

$$\begin{aligned} &\tilde{X}^T (P(A - LC) + (A - LC)^T P)\tilde{X} \\ &- \tilde{X}^T P \Delta(t) C \tilde{X} - \tilde{X}^T (\Delta(t) C)^T P \tilde{X} + \\ &+ 2\rho P \tilde{X}^T \tilde{X} + \tilde{X}^T P W + W^T P \tilde{X} < 0 \end{aligned} \quad (26)$$

Using Lemma 2 on the second and third term, with  $\varepsilon_1$  results in:

$$\begin{aligned} &\tilde{X}^T (P(A - LC) + (A - LC)^T P + \varepsilon_1^{-1} P P)\tilde{X} \\ &+ \varepsilon_1 \tilde{X}^T C^T [\Delta(t)]^T \Delta(t) C \tilde{X} + 2\rho P \tilde{X}^T \tilde{X} \\ &+ \tilde{X}^T P W + W^T P \tilde{X} < 0 \end{aligned} \quad (27)$$

Since  $\|\Delta(t)\| \leq r$  the inequality (27) can be written as:

$$\tilde{X}^T (M_{\tilde{X}}) \tilde{X} + 2\rho P \tilde{X}^T \tilde{X} + \tilde{X}^T P W + W^T P \tilde{X} < 0 \quad (28)$$

Where

$$M_{\tilde{X}} = P(A - LC) + (A - LC)^T P + \varepsilon_1^{-1} P P + \varepsilon_1 r^2 C^T C$$

Using Definition.2, when  $w = 0$  sufficient conditions for observer convergence can be obtained by setting:

$$\tilde{X}^T (M_{\tilde{X}} + 2\rho P) \tilde{X} < 0 \quad (29)$$

or equally:

$$M_{\tilde{X}} + 2\rho P < 0 \quad (30)$$

Using Definition.2 for the  $H_{\infty}$  observer we have:

$$\tilde{X}^T(t) \tilde{X}(t) - \eta W^T(t) W(t) \leq 0 \quad (31)$$

On the other hand, for the stability of the observer it is

necessary to have  $\frac{dV(t)}{dt} < 0$  which causes:

$$\tilde{X}^T \tilde{X} - \eta W^T W + \frac{dV(t)}{dt} < 0 \quad (32)$$

Then, using (21) and (28) is followed by:

$$\begin{aligned} \frac{dV(t)}{dt} &= \tilde{X}^T M_{\tilde{X}} \tilde{X} + 2\rho P \tilde{X}^T \tilde{X} + \tilde{X}^T P W \\ &+ W^T P \tilde{X} - 2 \int_0^{\infty} \omega \mu(\omega) z^T(\omega, t) P z(\omega, t) d\omega \end{aligned} \quad (33)$$

Inequality (32) can be rewritten as:

$$\begin{aligned} &\tilde{X}^T \tilde{X} - \eta w^T w + \tilde{X}^T M_{\tilde{X}} \tilde{X} + 2\rho P \tilde{X}^T \tilde{X} + \\ &- 2 \int_0^{\infty} \omega \mu(\omega) z^T(\omega, t) P z(\omega, t) d\omega \\ &+ \tilde{X}^T P W + W^T P \tilde{X} < 0 \end{aligned} \quad (34)$$

To summarize (34), a sufficient condition is:

$$\begin{aligned} &\tilde{X}^T (M_{\tilde{X}} + 2\rho P + I) \tilde{X} - \eta W^T W \\ &+ \tilde{X}^T P W + W^T P \tilde{X} < 0 \end{aligned} \quad (35)$$

which implies that:

$$X_w^T \begin{bmatrix} M_{\tilde{X}} + 2\rho P + I & P \\ P & -\eta I \end{bmatrix} X_w < 0 \quad (36)$$

with  $X_w = [\tilde{X} \ W]^T$ . This can be altered to an LMI

by using Schur complement as:

$$\begin{bmatrix} \Gamma & P & P \\ * & -\eta I & 0 \\ * & * & -\varepsilon_1 I \end{bmatrix} < 0 \quad (37)$$

with

$$\begin{aligned} \Gamma &= P(A - LC) + (A - LC)^T P + \varepsilon_1 r^2 C^T C + 2\rho P + I, \\ K_L &= PL \text{ and } P > 0. \square \end{aligned}$$

**Remark:** By considering  $W = 0$  in system (8), the error dynamic (14) changes to:

$${}_0D_t^q \tilde{X} = (A - LC - \Delta(t)C)\tilde{X} + \phi(X, U) - \phi(\hat{X}, U) \quad (38)$$

Using a continuous frequency distributed model and the Lyapunov function (18) leads the sufficient condition for observer convergence to (30), and the LMI approach to achieve the resilient observer gain alters to:

$$\begin{pmatrix} N_{OL} & P \\ P & -\varepsilon_1 I \end{pmatrix} < 0 \quad (39)$$

in which:

$$N_{OL} = P(A - LC) + (A - LC)^T P + \varepsilon_1 r^2 C^T C + 2\rho P$$

Due to Lemma.1 and inequality (36) a feasible solution for our main theorem will make (39) feasible but the reverse is not necessarily correct.

### III. Prior Lipschitz fractional order $H_\infty$ resilient observer design

As discussed in the introduction, several works have planned the observer design for Lipschitz fractional order systems. In this section, we will present the results of a recent and complete article on the Lipschitz fractional order  $H_\infty$  resilient observer design, which is very similar to the topic discussed in this article. This similarity will help us to have a better comparison.

Reference [6] has considered the non-fragile nonlinear observer (13) for system (8) while  $\phi(X, u)$  is a Lipschitz function with Lipschitz constant  $\gamma$ . This paper by using continuous frequency distribution has a comparison with [1, which has presented  $H_\infty$  resilient observer design for Lipschitz fractional order systems based on an iterative algorithm. [6] claims to have a bigger feasibility region for the designed observer besides simpler computing.

The Result of [6] is summarized in the following Lemma.

**Lemma 3:** Consider the resilient observer (13). This observer has a stable observation if there exist positive real numbers  $\varepsilon_1, \varepsilon_2$  and matrices  $P > 0$ , while the proportional observer gain is the solution of the following constrained LMI:

$$\begin{bmatrix} N_L + I & \frac{P}{2} & P^T & P^T \\ \frac{P^T}{2} & -\eta I & 0 & 0 \\ P & 0 & -2\varepsilon_1 I & 0 \\ P & 0 & 0 & -2\varepsilon_2 I \end{bmatrix} < 0 \quad (40)$$

in which

$$N_L = \frac{PA + A^T P^T}{2} - \frac{SC + C^T S^T}{2} + \frac{\varepsilon_2 r^2}{2} C^T C + \frac{\varepsilon_1}{2} \gamma^2 I,$$

and  $\eta$  is the  $L_2$  gain from disturbance to error as introduced in (4),  $\gamma > 0$  is Lipschitz constant of nonlinear function in system (8) while  $r$  is the known bound of the additive perturbation on the error gain.  $S = PL$  and the proportional observer gain is equal to  $L = P^{-1}S$ .

At first glance, Theorem 1 has less degrees and variables than Lemma 3. And at second glance, the constant of one-sided Lipschitz in (15) is of the first power while the constant of Lipschitz in (40) is of the second power.

Overall, it looks like having a bigger feasibility region for (15) versus (40), but to have a better intuition the simulation results will be shown in the next section.

### IV. Simulation Results

We have claimed that the proposed observer for one-sided Lipschitz has a more feasible region in comparison with Lipschitz systems. On the other hand, in [20] has been shown that any Lipschitz system is one-sided Lipschitz. In this section, we consider a financial model to show the effectiveness of the proposed observer besides having a comparison with the results of [6] as one of the similar articles with the approach of the present article in the field of Lipschitz nonlinear fractional order systems.

We introduce the fractional commensurate order financial system [2] that describes a fractional order model of three state variables  $x_1, x_2$  and  $x_3$  which stand for the interest rate, the investment demand, and the price index, respectively. The model is described by:

$$D^q X = \begin{bmatrix} -a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{bmatrix} X + \begin{bmatrix} x_1 x_2 \\ 1 - x_1^2 \\ 0 \end{bmatrix} + W \quad (41)$$

$$y = [1 \quad 1 \quad 0] X$$

where constant value  $a \geq 0$  is the saving amount, constant value  $b \geq 0$  is the cost per investment, and constant value  $c \geq 0$  is the elasticity of the demand of commercial markets.

In this paper, we consider  $q = 0.9, r = 1$  and  $\Delta(t) = [0.8, 0.5, 0.2]^T \sin(4t)$  where  $a = 0.5, b = 2$  and  $c = 3$ . As shown in [1], calculating the Lipschitz constant by using Jacobian matrix of nonlinear part

$\phi(X,U) = [x_1 x_2, 1-x_1^2, 0]^T$  and considering the states' initial value equal to  $(-0.8, -2, 1)$  results in  $\gamma = 0.26$ . Using (12) the equality of  $\rho$  and Lipschitz constant in Lipschitz systems is shown, which causes  $\rho = \gamma = 0.26$ . With the use of YALMIP toolbox [2] and LMI control toolbox [2] in MATLAB as solver, the feasible solution for (15) with  $\eta_{\min} = 0.6286$  is derived as:

$$S = [18258.02 \quad 18257.22 \quad -0.284]^T \quad \varepsilon_1 = 29919,$$

and 
$$P = \begin{bmatrix} 27.1687 & 26.7745 & 0.0446 \\ 26.7745 & 28.1085 & -0.6645 \\ 0.0446 & -0.6645 & 1.9451 \end{bmatrix}.$$

Therefore, the observer gain of the non-fragile observer is equal to  $L = [501.6269 \quad 172.8271 \quad 47.3926]^T$ .

To have a comparative study on Lipschitz and one-sided Lipschitz observer performance, solving (15) and (40) leads to the derivation of the non-fragile observer gain for system (41) in two ways. The feasible solution for (40) is derived as:  $\eta_{\min} = 0.6599$ ,  $\varepsilon_1 = 6.2491$ ,  $\varepsilon_2 = 26526$ ,

$$P = \begin{bmatrix} 55.3362 & 54.4146 & 0.0825 \\ 54.4146 & 56.9664 & -1.1589 \\ 0.0825 & -1.1589 & 3.7408 \end{bmatrix} \text{ and observer gain is equal to}$$

$$L = [271.8169 \quad 111.2596 \quad 28.2887]^T.$$

The results of the two theorem are similar, so in the continue, using the Ninteger toolbox [2] for simulating the fractional order dynamic consequences results in fig 1 and 2 which show the state estimation and observation errors for both Lipschitz and one-sided Lipschitz observer while input disturbance is introduced as

$$W = [0.5 \sin(3t), \frac{0.2}{t+1}, 0.01 \text{sgn}(\sin(\pi t))]^T.$$

Both observers work robustly and their performance is like each other. Estimation is good and acceptable while  $\eta \square 0.6$  and

$$W = [0.5 \sin(3t), \frac{0.2}{t+1}, 0.01 \text{sgn}(\sin(\pi t))]^T.$$

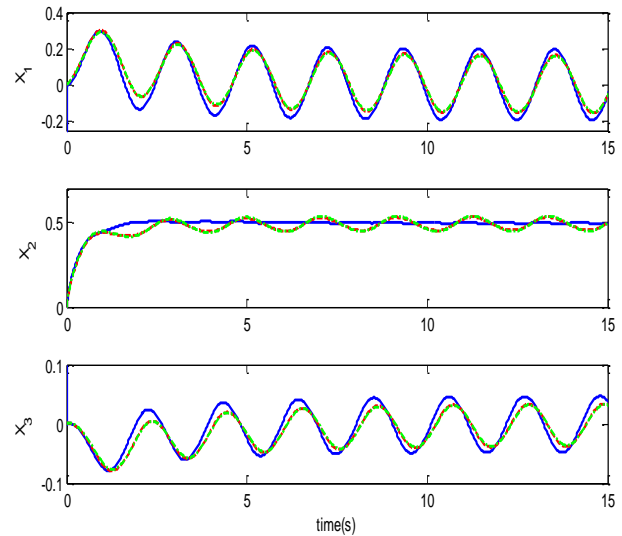


Fig. 1: Actual states (solid, blue), states estimation of the one-sided Lipschitz observer (dashed, red) and Lipschitz observer (dashed-dotted, green).

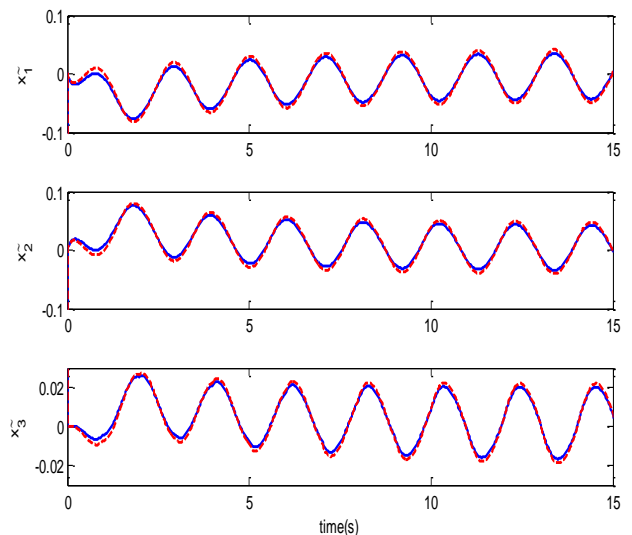
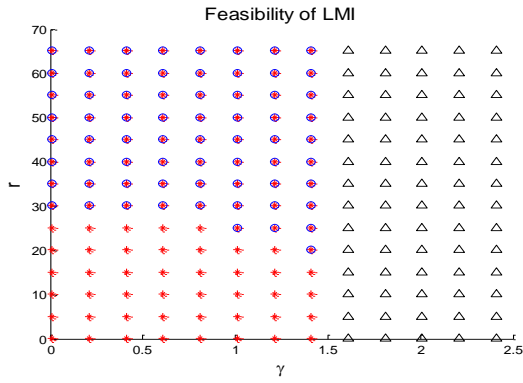
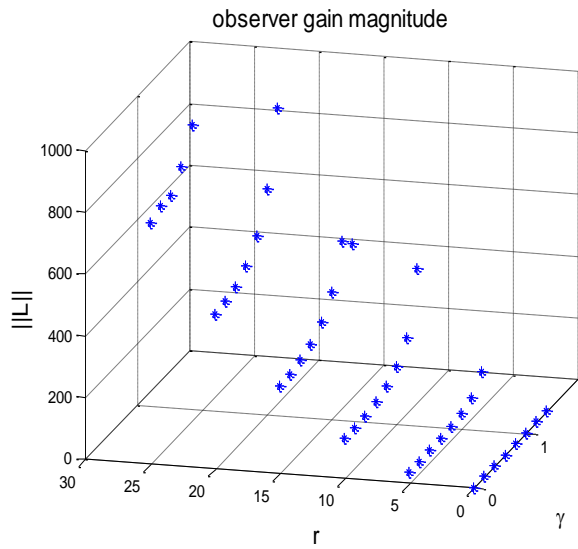


Fig. 2: States estimation errors of the one-sided Lipschitz observer (solid, blue) and Lipschitz observer (dashed, red).

In the next step, the feasibility region of (15) and (40) will be evaluated. Figure 3 shows the feasibility region of LMI (40) and  $\|L\|$  for different values of the Lipschitz constant and gain perturbation for the Lipschitz observer in lemma 3 while Fig 4 is the same for different one-sided Lipschitz constant and gain perturbation for LMI (15). It should be noted that value 1000 for  $\|L\|$  is an arbitrary boundary condition in accordance with the results obtained from the actual conditions of model (41) and some limitations in the physical realization.



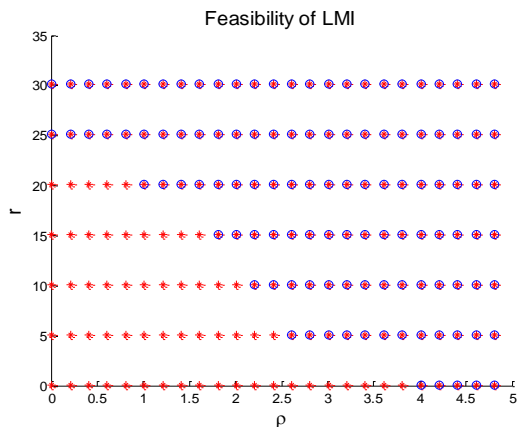
(a)



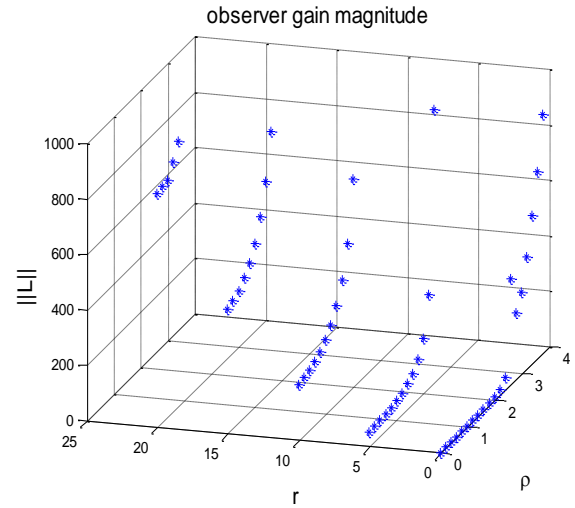
(b)

Fig.4: a) Feasibility region of (40) for different values of  $\gamma$  and  $r$ . (\* show the feasible solution,  $\Delta$  the infeasible solution and  $o$  shows the feasible observer gain that  $\|L\| > 1000$ ). b) Observer's gain magnitude ( $\|L\| < 1000$ )

It can be seen that inequality (40) becomes infeasible for  $\gamma > 1.5$  and the observer's gain will have large values ( $\|L\| > 1000$ ) by increasing  $r$ .



(a)



(b)

Fig.5: a) Feasibility region for the proposed method for different values of  $\rho$  and  $r$ . (\* show the feasible solution, + the infeasible solution and  $o$  shows observer gain more than 1000). b) Observer's gain magnitude ( $\|L\| < 1000$ )

Figure 5 illustrates the feasibility region of the one-sided Lipschitz observer including a wide range of values for  $\rho$  and  $r$ . However, increasing  $r$  and  $\rho$  will cause observer gain magnitude to become large ( $\|L\| > 1000$ ) but the first infeasible response will be obtained by  $\rho = 35$  and  $r = 30$ .

As it was expected and explained in section 4 of this paper, although LMI (15) and (40) will both lead to a robust and stable observer design, the feasibility region of inequality (15) is much larger than LMI (40) for Lipschitz systems. Besides LMI (15) comprises a wider range of observer design for both Lipschitz and one-sided Lipschitz systems.

## V. Conclusion

In this paper, we offered a systematic algorithm for designing a robust non-fragile fractional order observer for one-sided Lipschitz fractional order systems. Solving the resulting LMIs yields the observer gain, which ensures that state estimates converge to their true values besides minimizing the effects of exogenous disturbances and gain's perturbation. In addition, bigger feasibility region is shown for the proposed design in comparison with one of the latest and most similar observer designs for Lipschitz systems. The effectiveness of the proposed observer is investigated through the simulation of a fractional commensurate order financial system. Future studies can consider observer design for non-commensurate fractional order one-sided systems with simultaneous

time-delay, model uncertainty and exogenous disturbance input.

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