Delay-dependent $L_2 - L_\infty$ filter design for stochastic genetic regulatory networks in presence of time-varying delays

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Abstract

This paper addresses robust state estimation problem for Genetic Regulatory Networks (GRNs). A delay-dependent robust $L_2 - L_\infty$ filter is designed for a realistic nonlinear stochastic model of GRN. The model provided is the most complete model used in the literature so far, in the sense that delays are time-varying, parameter uncertainties (time-varying and norm-bounded) are considered, stochastic noises appear at the state equations as well as the measurement equations. Besides, stochastic noise and disturbance are considered simultaneously in this model. Using a proper Lyapunov-Krasovskii functional based on delay decomposition approach, sufficient conditions for the existence of the filter are derived in terms of linear matrix inequality (LMI). These conditions ensure robust asymptotic mean square stability of the filtering error dynamics with a prescribed $L_2 - L_\infty$ disturbance attenuation level. By use of delay decomposition approach and using a lemma containing a stochastic integral inequality, the obtained conditions are delay-dependent and have less conservativeness. The filter parameters are determined then, as the solution of another LMI. A simulation study is also given to show the effectiveness of the proposed filter design procedure.

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1. Introduction

In a living cell, two mechanisms are in action: genes encode proteins and some of proteins regulate gene expression either negatively or positively. These mechanisms construct a closed loop structure, which is called Genetic Regulatory Network (GRN). Gene expression consists of two main processes: “transcription” and “translation”; genes are transcribed into mRNAs under the control of some proteins and each mRNA molecule is translated to synthesis a protein. In recent two decades, a great deal of research has been done to propose a model for GRNs and make analysis on them [1].

Two approaches are proposed to make a mathematical model for GRNs: the discrete time approach used in models such as Boolean networks [2] and the continuous time approach using the differential equations [3], [4]. In the Boolean model, only two states, ON or OFF are used to express the activity of each gene, and a Boolean function of the states of other related genes determines the state of a gene [5]. In the differential equation model, the concentrations of gene products, such as mRNAs and proteins, are considered as the continuous state variables. Examining practical data, it seems that gene expression levels would better be modeled as continuous rather than discrete. Therefore, in recent years, differential equations have often been used to describe genetic networks [6], [7].

The time delay is a key factor affecting dynamics of gene expression. Mathematical models without considering time delays may give wrong predictions of the mRNA and protein concentrations [8]. So a complete model should certainly include a proper consideration of time delay. On the other hand, gene regulation is an intrinsically noisy process. In general, the noises appear in gene expression in one of the two ways, namely, intracellular noise and extracellular noise. The intracellular noises are due to the probabilistic chemical reactions, random births and deaths of individual molecules [9] and the extracellular noises are created because of environment fluctuations [10].

Furthermore, there are often some unavoidable uncertainties in modeling GRNs, which result from using an approximate model for simplicity, external perturbations, parameter fluctuations and data errors. It is very likely that the system parameters identified from experimental data may form an unknown but bonded time-varying function. Therefore, when investigating the dynamical behaviors of GRNs, the norm-bonded parameter uncertainties should also be taken into account.

In practice, for identifying genes of interest and designing drugs, biologists are interested in obtaining the steady-state values of the real network states based on the measurement data. Unfortunately, due to the time delay, noises and unavoidable uncertainties, the actual measurements are far from the true states. This leads researchers to use filter that is to estimate the GRN states such that the estimation error asymptotically converges to zero in the mean square sense in the presence of time delays, noises and uncertainties.

A robust filtering problem has been addressed in [11] for a linear GRN with stochastic noises, where the time delay has been ignored. In [12] the filtering issue has been investigated for nonlinear delayed GRNs with stochastic noises but without considering parameter uncertainties. In [13] the robust filter has been designed for GRNs with time-varying delays, where stochastic noises have been considered at both the state and measurement equations but regulation nonlinearities have been ignored. The state estimation for stochastic nonlinear uncertain GRNs has been addressed in [14] and [15] but stochastic noises have been considered only at the state equations and the delays have been assumed constant. On the other hand, in none of the mentioned references, disturbance has been considered in GRN model. In [3], [16] and [17] a stochastic nonlinear model has been developed for GRN under stochastic noises and disturbance simultaneously but filtering problem has not been studied.

For time-delay systems such as GRNs, the filter design methods can be classified into two categories: delay-dependent [18] and delay-independent [19]. Since delay-dependent methods depend on the amount of delay, they are less conservative than delay-independent ones. In [20] the H_∞ delay-independent filter has been designed for nonlinear uncertain GRN under stochastic noises and disturbance with time-varying delays and stochastic noises only at the state equations.

In the current paper, the nonlinear GRN model in the form of differential equations, provided in the literature has been made complete such that norm-bonded parameter uncertainties enter both the system and measurement matrices, delays are time-varying, stochastic noises appear at both the state and measurement equations and in addition, stochastic noises and disturbance are considered in the model simultaneously. The feedback regulation is described by a sector-like nonlinear function, the time-varying delays enter into both the translation process and the feedback regulation process, and the stochastic noise is a scalar Brownian motion. We aim to estimate the true concentrations of the mRNA and protein by designing a delay-dependent robust l_2-l_∞ filter.

Using delay decomposition approach [21], a Lyapunov-Krasovskii functional (LKF) is chosen. It has been showed that this approach leads to less conservative results [22]. Then a stochastic integral inequality is introduced [21]. Using the LKF, the integral inequality and free weighting matrix technique [21], [23], [24], delay-dependent sufficient conditions for the existence of robust L_2-L_∞ filter are derived. These conditions are in a linear matrix inequality (LMI) format and ensure that the filtering error dynamics is robustly asymptotically stable in the mean square with a prescribed L_2-L_∞ attenuation level.

The rest of this paper is organized as follows: in section 2 the model and preliminaries are provided. In section 3 the delay-dependent sufficient conditions for the existence of robust L_2-L_∞ filter are first obtained in the LMI format, and then filter parameters are determined in terms of the solutions to some LMIs. In section 4 a three-node GRN is presented to demonstrate the effectiveness of the proposed filter. Conclusions are finally given in section 5.

2. Model formulation and preliminaries

In this section, the proposed model will be provided later as equation (6). To introduce this model in some simple steps, first consider the nonlinear delayed GRN model with SUM regulatory functions [3] provided in [15]:

$$\begin{cases}\dot{m}(t)=-A_1m(t)+Bf(p(t-t_1))+B_0\\ \dot{p}(t)=-A_2p(t)+Dm(t-t_2)\end{cases}\tag{1}$$

where $m(t)$ equals $[m_1(t),m_2(t),...,m_n(t)]^T\in\mathbf{R}^n$ and $p(t)=[p_1(t),p_2(t),...,p_n(t)]^T\in\mathbf{R}^n$ in which $m_i(t)$ and $p_i(t),\ i=1,...,n$ are respectively the concentrations of mRNA and protein of the i^{th} node at time t for a GRN with n nodes; $A_1=diag\{a_{11},a_{12},...,a_{1n}\}$ and $A_2=diag\{c_{11},c_{12},...,c_{1n}\}$ where a_{1i} and $c_{1i},\ i=1,...,n$ denote, respectively, the degradation rates of mRNA and protein of the i^{th} node; $D=diag\{d_1,d_2,...,d_n\}$ where $d_i,\ i=1,...,n$ is the translation rate of the i^{th} node. $B=(b_{ij})\in\mathbf{R}^{n\times n}$ is the coupling matrix defined as follows:

$$b_{ij}=\begin{cases}a_{ij} & \textit{if transcription factor } j \textit{ is an activator of gene } i,\\ 0 & \textit{if there is no link from node } j \textit{ to } i,\\ -a_{ij} & \textit{if transcription factor } j \textit{ is a repressor of gene } i.\end{cases}\tag{2}$$

$f(p(t))$ equals $[f_1(p_1(t)),..., f_n(p_n(t))]^T$ where nonlinear function $f_j(.)$ represents the feedback regulation of the protein on the transcription. This function is supposed to be a monotonic function in Hill form [15], $f_j(x) = (x/b_j)^{H_j}/(1+(x/b_j)^{H_j})$ where H_j is the Hill coefficient and $b_j > 0$ is a constant. B_0 equals $[b_{01}, b_{02},..., b_{0n}]^T$ where $b_{0i} = \sum_{j\in I_i} a_{ij}$ in which I_i is the set of all *j* nodes which are repressors of gene *i*. $t_1 > 0$ denotes the translation delay and $t_2 > 0$ represents the feedback regulation delay.

Assume that vectors m^* and p^* are the equilibrium point vectors of (1). Shifting the state vectors to the origin, $x_m(t) = m(t) - m^*$, $x_p(t) = p(t) - p^*$ yields

$$\begin{cases} \dot{x}_m(t) = -A_1 x_m(t) + Bg(x_p(t-t_1)) \\ \dot{x}_p(t) = -A_2 x_p(t) + Dx_m(t-t_2) \end{cases} \tag{3}$$

where $g(x_p(t)) = [g_1(x_{p1}(t)),..., g_n(x_{pn}(t))]^T$ and $g_i(x_{pi}(t)) = f_i(x_{pi}(t) + p_i^*) - f_i(p_i^*)$.

Assumption 1: It is assumed that the function $g_i(.)$ satisfies the following sector condition

$$0 \le \frac{g_i(x_i)}{x_i} \le l_i,\ \forall x_i \in \mathbf{R},\ x_i \ne 0;\ g_i(0) = 0,\ i = 1,....,n \tag{4}$$

Considering $x_i > 0$ and (4) we can conclude $g_i(x_i) > 0$. So the sector condition (4) is equivalent to

$$g^T(x)(g(x) - Lx) \le 0, L = diag\{l_1, l_2,..., l_n\} > 0 \tag{5}$$

In this paper, we consider the following stochastic nonlinear model. This model is a completed version of the model provided in [12] such that delays are time-varying [13], stochastic noises appear at both the state dynamics and measurement equations [13], norm-bounded parameter uncertainties are taken into account [15] and this model is considered to have stochastic noises and disturbance simultaneously [3], [16], [17]:

$$\begin{cases} dx_m(t) = \\ \quad \left[-A_1(t)x_m(t) + B(t)g(x_p(t-t_1(t))) + A_{v1}(t)u(t)\right]dt \\ \quad +\left[E_1(t)x_m(t) + E_{v1}(t)u(t)\right]dw_1(t) \\ dx_p(t) = \\ \quad \left[-A_2(t)x_p(t) + D(t)x_m(t-t_2(t)) + A_{v2}(t)u(t)\right]dt \\ \quad +\left[F_1(t)x_p(t) + F_{v1}(t)u(t)\right]dw_2(t) \\ dy_m(t) = \left[C_1(t)x_m(t) + C_{v1}(t)u(t)\right]dt + \\ \qquad \left[E_2(t)x_m(t) + E_{v2}(t)u(t)\right]dw_1(t) \\ dy_p(t) = \left[C_2(t)x_p(t) + C_{v2}(t)u(t)\right]dt + \\ \qquad \left[F_2(t)x_p(t) + F_{v2}(t)u(t)\right]dw_2(t) \\ z_m(t) = H_m\, x_m(t) \\ z_p(t) = H_p\, x_p(t) \\ x_m(t) = f_m(t),\ x_p(t) = f_p(t),\ \forall t \in [-2t, 0] \end{cases} \tag{6}$$

where $y_m(t)$ equals $[y_{m1}(t), y_{m2}(t),..., y_{mr}(t)]^T \in \mathbf{R}^r$ and $y_p(t)$ equals $[y_{p1}(t), y_{p2}(t),..., y_{pr}(t)]^T \in \mathbf{R}^r$ in which $y_{mj}(t)$ and $y_{pj}(t)$ represent respectively the expression levels of mRNA and protein of the *j*th node at time *t*. $C_1(t)$ and $C_2(t)$ are the transformation matrices between the observation variables and the internal state variables. The time-varying scalars $t_1(t)$ and $t_2(t)$ satisfy $0 \le t_i(t) \le t_i$, $\dot{t}_i(t) \le d_i < 1$, $i = 1, 2$ and t is defined as $t = \max\{t_1, t_2\}$. $w_1(t)$ and $w_2(t)$ are scalar Brownian motions with zero mean and unit variance, which are mutually uncorrelated. $u(t) \in \mathbf{R}^m$ is the disturbance belonging to $L_2 - L_\infty$, i.e. the set of signals with bounded L_2 norm. The L_2 norm of $u(t)$ is defined as $\|u(t)\|_2 = \left\{\mathbf{E}\{\int_0^\infty u^T(t)u(t)dt\}\right\}^{1/2}$, where $\mathbf{E}$ denotes mathematical expectation.

$A_{v1}(t), A_{v2}(t), C_{v1}(t), C_{v2}(t), E_{v1}(t), F_{v1}(t), E_{v2}(t)$ and $F_{v2}(t)$ form the influence of disturbance on the states and measurements. $z_m(t) \in \mathbf{R}^q$ and $z_p(t) \in \mathbf{R}^q$ are the concentrations of mRNA and protein of nodes that we are interested. $f_m(t)$ and $f_p(t)$ are the initial condition functions of $x_m(t)$ and $x_p(t)$.

The time-varying matrices of system (6) have the following form:

$$\begin{aligned} &A_i(t) = A_i + \Delta A_i(t),\ \ A_{vi}(t) = A_{vi} + \Delta A_{vi}(t), \\ &E_i(t) = E_i + \Delta E_i(t),\ E_{vi}(t) = E_{vi} + \Delta E_{vi}(t) \\ &C_i(t) = C_i + \Delta C_i(t),\ C_{vi}(t) = C_{vi} + \Delta C_{vi}(t), \\ &F_i(t) = F_i + \Delta F_i(t),\ \ F_{vi}(t) = F_{vi} + \Delta F_{vi}(t) \\ &B(t) = B + \Delta B(t),\ \ D(t) = D + \Delta D(t)\ ,\ \ i = 1, 2 \end{aligned} \tag{7}$$

where $A_i, A_{vi}, E_i, E_{vi}, C_i, C_{vi}, F_i, F_{vi}, B$ and D are known real constant matrices with appropriate dimensions. $\Delta A_i(t), \Delta A_{vi}(t), \Delta E_i(t), \Delta E_{vi}(t), \Delta C_i(t), \Delta C_{vi}(t), \Delta F_i(t), \Delta F_{vi}(t), \Delta B(t)$ and $\Delta D(t)$ are unknown real time-varying matrices representing norm-bounded parameter uncertainties satisfying

$$\begin{bmatrix} \Delta A_1(t) & \Delta B(t) & \Delta A_{v1}(t) & \Delta E_1(t) & \Delta E_{v1}(t) \\ \Delta A_2(t) & \Delta D(t) & \Delta A_{v2}(t) & \Delta F_1(t) & \Delta F_{v1}(t) \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \Delta(t) \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & H_5 \end{bmatrix}$$
$$\begin{bmatrix} \Delta C_1(t) & \Delta C_{v1}(t) & \Delta E_2(t) & \Delta E_{v2}(t) \\ \Delta C_2(t) & \Delta C_{v2}(t) & \Delta F_2(t) & \Delta F_{v2}(t) \end{bmatrix} = \begin{bmatrix} G_3 \\ G_4 \end{bmatrix} \Delta(t) \begin{bmatrix} H_1 & H_3 & H_4 & H_5 \end{bmatrix} \quad (8)$$

where $G_j\,(j=1,2,3,4)$ and $H_j\,(j=1,2,3,4,5)$ are known real constant matrices and $\Delta(t)$ is the unknown, time-varying, matrix valued, Lebesgue measurable function satisfying

$$\Delta^T(t)\Delta(t) \le I, \quad \forall t \ge 0 \quad (9)$$

The main aim of this paper is to estimate the concentration of mRNA and protein, i.e. $x_m(t)$ and $x_p(t)$ at (6) through their measurement output, i.e. $y_m(t)$ and $y_p(t)$. The linear filter considered here is of the following form

$$\begin{cases} d\hat{x}_m(t) = A_f \hat{x}_m(t)\,dt + B_f\,dy_m(t) \\ d\hat{x}_p(t) = C_f \hat{x}_p(t)\,dt + D_f\,dy_p(t) \\ \hat{z}_m(t) = H_{mf}\, \hat{x}_m(t) \\ \hat{z}_p(t) = H_{pf}\, \hat{x}_p(t) \\ \hat{x}_m(t) = y_m(t),\ \hat{x}_p(t) = y_p(t),\ \forall t \in [-2t, 0] \end{cases} \quad (10)$$

where $\hat{x}_m(t) \in \mathbf{R}^n$ and $\hat{x}_p(t) \in \mathbf{R}^n$ are the estimates for $x_m(t)$ and $x_p(t)$, respectively; $y_m(t)$ and $y_p(t)$ are the initial functions of $\hat{x}_m(t)$ and $\hat{x}_p(t)$, respectively; and $A_f, B_f, C_f, D_f, H_{mf}$ and H_{pf} are appropriately dimensioned filter parameter matrices to be determined.

Defining

$$\bar{x}_m(t) := \begin{bmatrix} x_m(t) \\ \hat{x}_m(t) \end{bmatrix},\ \bar{x}_{mt_2} := \begin{bmatrix} x_m(t-t_2(t)) \\ \hat{x}_m(t-t_2(t)) \end{bmatrix},$$
$$\bar{x}_p(t) := \begin{bmatrix} x_p(t) \\ \hat{x}_p(t) \end{bmatrix},\ \bar{x}_{pt_1} := \begin{bmatrix} x_p(t-t_1(t)) \\ \hat{x}_p(t-t_1(t)) \end{bmatrix}$$
$$e_m(t) = z_m(t) - \hat{z}_m(t),\ e_p(t) = z_p(t) - \hat{z}_p(t) \quad (11)$$
$$\bar{x}(t) := \begin{bmatrix} \bar{x}_m^T(t) & \bar{x}_p^T(t) \end{bmatrix}^T,$$
$$r(t) := \begin{bmatrix} f_m^T(t) & y_m^T(t) & f_p^T(t) & y_p^T(t) \end{bmatrix}^T$$

and augmenting the states of the filter (10) to the model of (6), the following filtering error dynamics is obtained:

$$\begin{cases} d\bar{x}_m(t) = \left[\bar{A}(t)\bar{x}_m(t) + \bar{B}(t)\,g(K\bar{x}_{pt_1}) + \bar{A}_{v1}(t)u(t)\right]dt \\ \qquad + \left(\bar{E}(t)K\bar{x}_m(t) + \bar{E}_v(t)u(t)\right)dw_1(t) \\ d\bar{x}_p(t) = \left[\bar{C}(t)\bar{x}_p(t) + \bar{D}(t)K\bar{x}_{mt_2} + \bar{A}_{v2}(t)u(t)\right]dt \\ \qquad + \left(\bar{F}(t)K\bar{x}_p(t) + \bar{F}_v(t)u(t)\right)dw_2(t) \\ e_m(t) = \bar{H}_m\,\bar{x}_m(t) \\ e_p(t) = \bar{H}_p\,\bar{x}_p(t) \\ \bar{x}(t) = r(t),\ \forall t \in [-2t, 0] \end{cases} \quad (12)$$

where

$$\begin{aligned} &\bar{A}(t) = \bar{A} + \Delta\bar{A}(t),\ \bar{B}(t) = \bar{B} + \Delta\bar{B}(t), \\ &\bar{E}(t) = \bar{E} + \Delta\bar{E}(t),\ \bar{E}_v(t) = \bar{E}_v + \Delta\bar{E}_v(t) \\ &\bar{C}(t) = \bar{C} + \Delta\bar{C}(t),\ \bar{D}(t) = \bar{D} + \Delta\bar{D}(t), \\ &\bar{F}(t) = \bar{F} + \Delta\bar{F}(t),\ \bar{F}_v(t) = \bar{F}_v + \Delta\bar{F}_v(t) \\ &\bar{A}_{vi}(t) = \bar{A}_{vi} + \Delta\bar{A}_{vi}(t),\ i = 1,2 \end{aligned} \quad (13)$$

in which

$$\begin{aligned} &\bar{A} = \begin{bmatrix} -A_1 & 0 \\ B_f C_1 & A_f \end{bmatrix},\ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \bar{A}_{v1} = \begin{bmatrix} A_{v1} \\ B_f C_{v1} \end{bmatrix}, \\ &\bar{E} = \begin{bmatrix} E_1 \\ B_f E_2 \end{bmatrix},\ \bar{E}_v = \begin{bmatrix} E_{v1} \\ B_f E_{v2} \end{bmatrix} \\ &\bar{C} = \begin{bmatrix} -A_2 & 0 \\ D_f C_2 & C_f \end{bmatrix},\ \bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix},\ \bar{A}_{v2} = \begin{bmatrix} A_{v2} \\ D_f C_{v2} \end{bmatrix}, \\ &\bar{F} = \begin{bmatrix} F_1 \\ D_f F_2 \end{bmatrix},\ \bar{F}_v = \begin{bmatrix} F_{v1} \\ D_f F_{v2} \end{bmatrix} \\ &\bar{H}_m = \begin{bmatrix} H_m & -H_{mf} \end{bmatrix},\ \bar{H}_p = \begin{bmatrix} H_p & -H_{pf} \end{bmatrix},\ K = \begin{bmatrix} I & 0 \end{bmatrix} \\ &\Delta\bar{A}(t) = \bar{G}_1\Delta(t)\bar{H}_1,\ \Delta\bar{B}(t) = \bar{G}_2\Delta(t)H_2, \\ &\Delta\bar{E}(t) = \bar{G}_3\Delta(t)H_4,\ \Delta\bar{E}_v(t) = \bar{G}_3\Delta(t)H_5 \\ &\Delta\bar{C}(t) = \bar{G}_4\Delta(t)\bar{H}_1,\ \Delta\bar{D}(t) = \bar{G}_5\Delta(t)H_2, \\ &\Delta\bar{F}(t) = \bar{G}_6\Delta(t)H_4,\ \Delta\bar{F}_v(t) = \bar{G}_6\Delta(t)H_5 \\ &\Delta\bar{A}_{v1}(t) = \bar{G}_3\Delta(t)H_3,\ \Delta\bar{A}_{v2}(t) = \bar{G}_6\Delta(t)H_3 \\ &\bar{G}_1 = \begin{bmatrix} -G_1 \\ B_f G_3 \end{bmatrix},\ \bar{G}_2 = \begin{bmatrix} G_1 \\ 0 \end{bmatrix},\ \bar{G}_3 = \begin{bmatrix} G_1 \\ B_f G_3 \end{bmatrix},\ \bar{H}_1 = \begin{bmatrix} H_1 & 0 \end{bmatrix} \\ &\bar{G}_4 = \begin{bmatrix} -G_2 \\ D_f G_4 \end{bmatrix},\ \bar{G}_5 = \begin{bmatrix} G_2 \\ 0 \end{bmatrix},\ \bar{G}_6 = \begin{bmatrix} G_2 \\ D_f G_4 \end{bmatrix} \end{aligned} \quad (14)$$

To investigate the convergence of the filtering error dynamics, the following stability definitions must be considered:

Definition 1 [20]: A system described by equation (6) with $u(t) = 0$ is said to be *robustly mean square stable* for all admissible uncertainties (8)-(9), if for any scalar $e > 0$ there exists a scalar $d(e) > 0$ such that $\mathbf{E}\left\{|x_m(t)|^2\right\} < e$ and $\mathbf{E}\left\{|x_p(t)|^2\right\} < e$, $\forall t > 0$ when $\sup_{t\in[-2t,0]} \mathbf{E}\left\{|f_m(t)|^2\right\} < d(e)$ and $\sup_{t\in[-2t,0]} \mathbf{E}\left\{|f_p(t)|^2\right\} < d(e)$. In addition, if

$\lim_{t\to\infty}\mathbf{E}\left\{\left|x_m(t)\right|^2+\left|x_p(t)\right|^2\right\}=0$ for any initial conditions, then this system is said to be robustly *asymptotically* mean square stable.

Definition 2 [19]: Given a scalar $g>0$, the filtering error dynamics (12) is said to be asymptotically mean square stable *with the* L_2-L_∞ *attenuation level* g, if it is asymptotically mean square stable with $u(t)=0$, and under zero initial condition, it satisfies $\left\|e_m(t)\right\|_{E_\infty}<g\left\|u(t)\right\|_2$ and $\left\|e_p(t)\right\|_{E_\infty}<g\left\|u(t)\right\|_2$ for all nonzero $u(t)\in L_2[0,\infty)$, where $\|.\|_{E_\infty}$ for the signal $e(t)$ is defined as $\left\|e(t)\right\|_{E_\infty}=\sup_t\sqrt{\mathbf{E}\left\{\left|e(t)\right|^2\right\}}$.

Assumption 2 [21]: The system (6) with $u(t)=0$ is robustly asymptotically mean square stable.

Lemma 1 [19]: Given appropriately dimensioned matrices $\Sigma_1,\Sigma_2,\Sigma_3$ with $\Sigma_1^T=\Sigma_1$, then $\Sigma_1+\Sigma_3\Delta(t)\Sigma_2+\Sigma_2^T\Delta^T(t)\Sigma_3^T<0$ holds for all $\Delta(t)$ satisfying $\Delta^T(t)\Delta(t)\le I$ if and only if for some $e>0,\ \Sigma_1+e^{-1}\Sigma_3\Sigma_3^T+e\,\Sigma_2^T\Sigma_2<0$.

Lemma 2 [21]: Let ***n***-dimensional vector functions $x(t)$, $j(t)$ and $g(t)$ satisfy the stochastic differential equation $dx(t)=j(t)\,dt+g(t)\,dw(t)$ where $w(t)$ is a one-dimensional Brownian motion. For any constant matrix $Z\ge 0\in\mathbf{R}^{n\times n}$ and scalar $h>0$, if the following integration is well defined, then the following inequality holds on it

$$-h\int_{t-h}^{t}j^T(s)Zj(s)\,ds\le\begin{bmatrix}x(t)\\ x(t-h)\end{bmatrix}^T\begin{bmatrix}-Z & Z\\ Z & -Z\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-h)\end{bmatrix}+2\begin{bmatrix}x(t)\\ x(t-h)\end{bmatrix}^T\begin{bmatrix}Z\\ -Z\end{bmatrix}\int_{t-h}^{t}g(s)\,dw(s)\qquad(15)$$

Proof: see [21].

The purpose of this paper is to design the robust L_2-L_∞ filter of the form (10) for the system (6) such that, for all stochastic noises and disturbance, admissible nonlinearities, uncertainties and time delays, the filtering error dynamics (12) is robustly asymptotically mean square stable with a prescribed L_2-L_∞ attenuation level g.

3. Robust L_2-L_∞ filter analysis and design

In this section, we first assume that the filter is designed and its matrices are exactly known in order to study the stability of the filter, i.e. convergence of the filtering error to zero. This study is based on the idea of *delay decomposition*, resulting to the delay-dependent conditions under which the filtering error is robustly asymptotically mean square stable with an L_2-L_∞ attenuation level g. Delay decomposition technique is to partition the time delay with an integer r. The parameter r is the number of delay partitions and the conservatism reduces as r increases [25].

Theorem 1: Given an integer $r\ge 1$, a scalar $g>0$ and filter parameters A_f,B_f,C_f,D_f,H_{mf} and H_{pf}. The filtering error dynamics (12) is robustly asymptotically mean square stable with the L_2-L_∞ attenuation level g**,** if there exist scalars $e_j>0,\ j=1,...,7$, matrices $P_i>0\in\mathbf{R}^{2n\times 2n},i=1,2$ and $n\times n$ matrices $R_i\ge 0,i=1,2$; $Q_{km}\ge 0$; $Q_{kp}\ge 0$; $Z_{km}\ge 0$; $Z_{kp}\ge 0,k=1,2,...,r$; S_1 and S_2 such that the following LMIs hold

$$\Xi=\begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & \Xi_{19} & \Xi_{110} & \Xi_{111}\\
* & \Xi_{22} & \bar{E}_v^TP_1 & \bar{F}_v^TP_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & -P_1 & 0 & 0 & 0 & P_1\bar{G}_3 & 0 & 0 & 0 & 0\\
* & * & * & -P_2 & 0 & 0 & 0 & 0 & 0 & P_2\bar{G}_6 & 0\\
* & * & * & * & -e_1I & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -e_2I & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -e_3I & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & -e_4I & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -e_5I & 0 & 0\\
* & * & * & * & * & * & * & * & * & -e_6I & 0\\
* & * & * & * & * & * & * & * & * & * & -e_7I
\end{bmatrix}<0\qquad(16)$$

$$\Omega_1=\begin{bmatrix}P_1 & \bar{H}_m^T\\ * & g^2I\end{bmatrix}>0,\ \Omega_2=\begin{bmatrix}P_2 & \bar{H}_p^T\\ * & g^2I\end{bmatrix}>0\qquad(17)$$

where

$$\Xi_{11}=\begin{bmatrix}
\Gamma_1 & 0 & 0 & 0 & P_1\bar{B} & \Delta_1 & 0 & 0 & 0 & \cdots & 0 & 0 & \Delta_5 & 0\\
* & \Gamma_2 & P_2\bar{D} & 0 & 0 & 0 & \Delta_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \Delta_6\\
* & * & \Gamma_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \Delta_7\\
* & * & * & \Gamma_4 & L & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
* & * & * & * & \Gamma_5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \Delta_8 & 0\\
* & * & * & * & * & \Gamma_{2m} & 0 & Z_{2m} & 0 & \cdots & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & \Gamma_{2p} & 0 & Z_{2p} & \ddots & \vdots & 0 & 0 & 0\\
* & * & * & * & * & * & * & \ddots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \Gamma_{rm} & 0 & Z_{rm} & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & \Gamma_{rp} & 0 & Z_{rp} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & \Gamma_{(r+1)m} & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & \Gamma_{(r+1)p} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & \Delta_3 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & * & \Delta_4
\end{bmatrix} \quad (18)$$

In which

$$\Xi_{12}=\left[\bar{A}_{v1}^T P_1+e_3 H_5^T H_4 K \quad \bar{A}_{v2}^T P_2+e_6 H_5^T H_4 K \quad 0 \quad \ldots \quad 0 \quad A_{v1}^T S_1 \quad A_{v2}^T S_2\right]^T$$

$$\Xi_{13}=\left[P_1\bar{E}K \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0\right]^T,$$

$$\Xi_{15}=\left[\bar{G}_1^T P_1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad \bar{G}_1^T K^T S_1 \quad 0\right]^T$$

$$\Xi_{14}=\left[0 \quad P_2\bar{F}K \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0\right]^T,$$

$$\Xi_{16}=\left[\bar{G}_2^T P_1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad \bar{G}_2^T K^T S_1 \quad 0\right]^T \quad (19)$$

$$\Xi_{17}=\left[0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0\right]^T,$$

$$\Xi_{18}=\left[0 \quad \bar{G}_4^T P_2 \quad 0 \quad \cdots \quad 0 \quad 0 \quad \bar{G}_4^T K^T S_2\right]^T$$

$$\Xi_{110}=\left[0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0\right]^T,$$

$$\Xi_{19}=\left[0 \quad \bar{G}_5^T P_2 \quad 0 \quad \cdots \quad 0 \quad 0 \quad \bar{G}_5^T K^T S_2\right]^T$$

$$\Xi_{111}=\left[\bar{G}_3^T P_1 \quad \bar{G}_6^T P_2 \quad 0 \quad \cdots \quad 0 \quad \bar{G}_3^T K^T S_1 \quad \bar{G}_6^T K^T S_2\right]^T$$

$$\Xi_{22}=-I+(e_3+e_6)H_5^T H_5+e_7 H_3^T H_3$$

And

$$\Gamma_1=P_1\bar{A}+\bar{A}^T P_1+K^T R_1 K+K^T Q_{1m} K - K^T Z_{1m} K+e_1\bar{H}_1^T\bar{H}_1+e_3 K^T H_4^T H_4 K$$

$$\Gamma_2=P_2\bar{C}+\bar{C}^T P_2+K^T R_2 K+K^T Q_{1p} K - K^T Z_{1p} K+e_4\bar{H}_1^T\bar{H}_1+e_6 K^T H_4^T H_4 K$$

$$\Gamma_3=(d_2-1)R_1+e_5 H_2^T H_2,\ \Gamma_4=(d_1-1)R_2,\Gamma_5=-2I+e_2 H_2^T H_2$$

$$\Gamma_{im}=Q_{im}-Q_{(i-1)m}-Z_{im}-Z_{(i-1)m},i=2,\ldots,r,\quad \Gamma_{(r+1)m}=-Z_{rm}-Q_{rm} \quad (20)$$

$$\Gamma_{ip}=Q_{ip}-Q_{(i-1)p}-Z_{ip}-Z_{(i-1)p},i=2,\ldots,r,\ \Gamma_{(r+1)p}=-Z_{rp}-Q_{rp}$$

$$\Delta_1=K^T Z_{1m},\ \Delta_3=h_2^2\sum_{k=1}^{r} Z_{km}-S_1-S_1^T,$$

$$\Delta_5=\bar{A}^T K^T S_1,\ \Delta_7=\bar{D}^T K^T S_2$$

$$\Delta_2=K^T Z_{1p},\ \Delta_4=h_1^2\sum_{k=1}^{r} Z_{kp}-S_2-S_2^T,$$

$$\Delta_6=\bar{C}^T K^T S_2,\ \Delta_8=\bar{B}^T K^T S_1$$

Proof: For presentation convenience, we let

$$\begin{aligned} x_m(t)&=\bar{A}(t)\bar{x}_m(t)+\bar{B}(t)\,g(K\bar{x}_{pt_1})+\bar{A}_{v1}(t)u(t)\\ x_p(t)&=\bar{C}(t)\bar{x}_p(t)+\bar{D}(t)K\bar{x}_{mt_2}+\bar{A}_{v2}(t)u(t)\end{aligned} \quad (21)$$

Then the filtering error dynamics (12) can be rewritten as

$$d\bar{x}_m(t)=x_m(t)\,dt+\left(\bar{E}(t)K\bar{x}_m(t)+\bar{E}_v(t)u(t)\right)dw_1(t) \quad (22)$$

$$d\bar{x}_p(t)=x_p(t)\,dt+\left(\bar{F}(t)K\bar{x}_p(t)+\bar{F}_v(t)u(t)\right)dw_2(t) \quad (23)$$

Using delay decomposition approach [21], we choose the following Lyapunov-Krasovskii functional candidate for the system (12):

$$V(\bar{x}(t),t)=V_1(\bar{x}(t),t)+V_2(\bar{x}(t),t)+V_3(\bar{x}(t),t)+V_4(\bar{x}(t),t) \quad (24)$$

where

$$V_1(\bar{x}(t),t)=\bar{x}_m^T(t)P_1\bar{x}_m(t)+\bar{x}_p^T(t)P_2\bar{x}_p(t) \quad (25)$$

$$V_2(\bar{x}(t),t)=\int_{t-t_2(t)}^{t}\bar{x}_m^T(s)K^TR_1K\bar{x}_m(s)ds+\int_{t-t_1(t)}^{t}\bar{x}_p^T(s)K^TR_2K\bar{x}_p(s)ds \tag{26}$$

$$V_3(\bar{x}(t),t)=\sum_{k=1}^{r}\int_{t-kh_2}^{t-(k-1)h_2}\bar{x}_m^T(s)K^TQ_{km}K\bar{x}_m(s)ds+\sum_{k=1}^{r}\int_{t-kh_1}^{t-(k-1)h_1}\bar{x}_p^T(s)K^TQ_{kp}K\bar{x}_p(s)ds \tag{27}$$

$$V_4(\bar{x}(t),t)=\sum_{k=1}^{r}\int_{-kh_2}^{-(k-1)h_2}\int_{t+q}^{t}x_m^T(s)(h_2K^TZ_{km}K)x_m(s)dsdq+\sum_{k=1}^{r}\int_{-kh_1}^{-(k-1)h_1}\int_{t+q}^{t}x_p^T(s)(h_1K^TZ_{kp}K)x_p(s)dsdq \tag{28}$$

where $h_i=t_i/r, i=1,2$. By Ito's differential formula [26], the stochastic differential of (24) along the filtering error dynamics (12) is

$$\begin{aligned}dV(\bar{x}(t),t)&=\mathcal{L}V(\bar{x}(t),t)dt\\&+2\bar{x}_m^T(t)P_1\left(\bar{E}(t)K\bar{x}_m(t)+\bar{E}_v(t)u(t)\right)dw_1(t)\\&+2\bar{x}_p^T(t)P_2\left(\bar{F}(t)K\bar{x}_p(t)+\bar{F}_v(t)u(t)\right)dw_2(t)\end{aligned} \tag{29}$$

where

$$\mathcal{L}V(\bar{x}(t),t)=\mathcal{L}V_1(\bar{x}(t),t)+\mathcal{L}V_2(\bar{x}(t),t)+\mathcal{L}V_3(\bar{x}(t),t)+\mathcal{L}V_4(\bar{x}(t),t) \tag{30}$$

and

$$\begin{aligned}\mathcal{L}V_1(\bar{x}(t),t)=&\\&2\bar{x}_m^T(t)P_1x_m(t)+2\bar{x}_p^T(t)P_2x_p(t)\\&+\left(\bar{x}_m^T(t)K^T\bar{E}^T(t)+u^T(t)\bar{E}_v^T(t)\right)P_1\left(\bar{E}(t)K\bar{x}_m(t)+\bar{E}_v(t)u(t)\right)\\&+\left(\bar{x}_p^T(t)K^T\bar{F}^T(t)+u^T(t)\bar{F}_v^T(t)\right)P_2\left(\bar{F}(t)K\bar{x}_p(t)+\bar{F}_v(t)u(t)\right)\end{aligned} \tag{31}$$

$$\begin{aligned}\mathcal{L}V_2(\bar{x}(t),t)=&\bar{x}_m^T(t)K^TR_1K\bar{x}_m(t)-(1-\dot{t}_2(t))\bar{x}_{mt_2}^TK^TR_1K\bar{x}_{mt_2}\\&+\bar{x}_p^T(t)K^TR_2K\bar{x}_p(t)-(1-\dot{t}_1(t))\bar{x}_{pt_1}^TK^TR_2K\bar{x}_{pt_1}\end{aligned} \tag{32}$$

$$\begin{aligned}\mathcal{L}V_3(\bar{x}(t),t)=&\\&\sum_{k=1}^{r}\bar{x}_m^T(t-(k-1)h_2)K^TQ_{km}K\bar{x}_m(t-(k-1)h_2)\\&-\sum_{k=1}^{r}\bar{x}_m^T(t-kh_2)K^TQ_{km}K\bar{x}_m(t-kh_2)\\&+\sum_{k=1}^{r}\bar{x}_p^T(t-(k-1)h_1)K^TQ_{kp}K\bar{x}_p(t-(k-1)h_1)\\&-\sum_{k=1}^{r}\bar{x}_p^T(t-kh_1)K^TQ_{kp}K\bar{x}_p(t-kh_1)\end{aligned} \tag{33}$$

$$\begin{aligned}\mathcal{L}V_4(\bar{x}(t),t)=&\sum_{k=1}^{r}x_m^T(t)(h_2^2K^TZ_{km}K)x_m(t)\\&-h_2\sum_{k=1}^{r}\int_{t-kh_2}^{t-(k-1)h_2}x_m^T(s)(K^TZ_{km}K)x_m(s)ds\\&+\sum_{k=1}^{r}x_p^T(t)(h_1^2K^TZ_{kp}K)x_p(t)\\&-h_1\sum_{k=1}^{r}\int_{t-kh_1}^{t-(k-1)h_1}x_p^T(s)(K^TZ_{kp}K)x_p(s)ds\end{aligned} \tag{34}$$

Applying Lemma 2 to (22), we obtain

$$\begin{aligned}&-h_2\sum_{k=2}^{r}\int_{t-kh_2}^{t-(k-1)h_2}x_m^T(s)(K^TZ_{km}K)x_m(s)ds\le\\&\sum_{k=2}^{r}\begin{bmatrix}K\bar{x}_m(t-(k-1)h_2)\\K\bar{x}_m(t-kh_2)\end{bmatrix}^T\begin{bmatrix}-Z_{km}&Z_{km}\\Z_{km}&-Z_{km}\end{bmatrix}\begin{bmatrix}K\bar{x}_m(t-(k-1)h_2)\\K\bar{x}_m(t-kh_2)\end{bmatrix}\\&+2\sum_{k=2}^{r}\begin{bmatrix}K\bar{x}_m(t-(k-1)h_2)\\K\bar{x}_m(t-kh_2)\end{bmatrix}^T\begin{bmatrix}Z_{km}\\-Z_{km}\end{bmatrix}K\\&\times\int_{t-kh_2}^{t-(k-1)h_2}\left(\bar{E}(s)K\bar{x}_m(s)+\bar{E}_v(s)u(s)\right)dw_1(s)\end{aligned} \tag{35}$$

$$\begin{aligned}&-h_2\int_{t-h_2}^{t}x_m^T(s)(K^TZ_{1m}K)x_m(s)ds\le\\&\begin{bmatrix}\bar{x}_m(t)\\K\bar{x}_m(t-h_2)\end{bmatrix}^T\begin{bmatrix}-K^TZ_{1m}K&K^TZ_{1m}\\Z_{1m}K&-Z_{1m}\end{bmatrix}\begin{bmatrix}\bar{x}_m(t)\\K\bar{x}_m(t-h_2)\end{bmatrix}\\&+2\begin{bmatrix}K\bar{x}_m(t)\\K\bar{x}_m(t-h_2)\end{bmatrix}^T\begin{bmatrix}Z_{1m}\\-Z_{1m}\end{bmatrix}K\int_{t-h_2}^{t}\left(\bar{E}(s)K\bar{x}_m(s)+\bar{E}_v(s)u(s)\right)dw_1(s)\end{aligned} \tag{36}$$

In a similar way, applying Lemma 2 to (23) yields

$$\begin{aligned}&-h_1\sum_{k=2}^{r}\int_{t-kh_1}^{t-(k-1)h_1}x_p^T(s)(K^TZ_{kp}K)x_p(s)ds\le\\&\sum_{k=2}^{r}\begin{bmatrix}K\bar{x}_p(t-(k-1)h_1)\\K\bar{x}_p(t-kh_1)\end{bmatrix}^T\begin{bmatrix}-Z_{kp}&Z_{kp}\\Z_{kp}&-Z_{kp}\end{bmatrix}\begin{bmatrix}K\bar{x}_p(t-(k-1)h_1)\\K\bar{x}_p(t-kh_1)\end{bmatrix}\\&+2\sum_{k=2}^{r}\begin{bmatrix}K\bar{x}_p(t-(k-1)h_1)\\K\bar{x}_p(t-kh_1)\end{bmatrix}^T\begin{bmatrix}Z_{kp}\\-Z_{kp}\end{bmatrix}K\\&\times\int_{t-kh_1}^{t-(k-1)h_1}\left(\bar{F}(s)K\bar{x}_p(s)+\bar{F}_v(s)u(s)\right)dw_2(s)\end{aligned} \tag{37}$$

$$\begin{aligned}&-h_1\int_{t-h_1}^{t}x_p^T(s)(K^TZ_{1p}K)x_p(s)ds\le\\&\begin{bmatrix}\bar{x}_p(t)\\K\bar{x}_p(t-h_1)\end{bmatrix}^T\begin{bmatrix}-K^TZ_{1p}K&K^TZ_{1p}\\Z_{1p}K&-Z_{1p}\end{bmatrix}\begin{bmatrix}\bar{x}_p(t)\\K\bar{x}_p(t-h_1)\end{bmatrix}\\&+2\begin{bmatrix}K\bar{x}_p(t)\\K\bar{x}_p(t-h_1)\end{bmatrix}^T\begin{bmatrix}Z_{1p}\\-Z_{1p}\end{bmatrix}K\int_{t-h_1}^{t}\left(\bar{F}(s)K\bar{x}_p(s)+\bar{F}_v(s)u(s)\right)dw_2(s)\end{aligned} \tag{38}$$

By using free weighting matrix technique [23], [24] and according to (21), for any $S_i\in\mathbf{R}^{n\times n}, i=1,2$ we have

$$2x_m^T(t)K^TS_1^TK\left\{\bar{A}(t)\bar{x}_m(t)+\bar{B}(t)g(K\bar{x}_{pt1})+\bar{A}_{v1}(t)u(t)-x_m(t)\right\}=0$$
$$2x_p^T(t)K^TS_2^TK\left\{\bar{C}(t)\bar{x}_p(t)+\bar{D}(t)K\bar{x}_{mt2}+\bar{A}_{v2}(t)u(t)-x_p(t)\right\}=0 \quad (39)$$

Since $K\bar{x}_{pt_1}=x_p(t-t_1(t))>0$ and noting (5) we have

$$-2g^T(K\bar{x}_{pt_1})(g(K\bar{x}_{pt_1})-LK\bar{x}_{pt_1})\geq 0 \quad (40)$$

It follows from (30), (39) and (40) that

$$\begin{aligned}\mathcal{L}V(\bar{x}(t),t)\leq\ & \mathcal{L}V_1(\bar{x}(t),t)+\mathcal{L}V_2(\bar{x}(t),t)+\mathcal{L}V_3(\bar{x}(t),t)+\mathcal{L}V_4(\bar{x}(t),t)\\ &+2x_m^T(t)K^TS_1^TK\left\{\bar{A}(t)\bar{x}_m(t)+\bar{B}(t)g(K\bar{x}_{pt1})+\bar{A}_{v1}(t)u(t)-x_m(t)\right\}\\ &+2x_p^T(t)K^TS_2^TK\left\{\bar{C}(t)\bar{x}_p(t)+\bar{D}(t)K\bar{x}_{mt2}+\bar{A}_{v2}(t)u(t)-x_p(t)\right\}\\ &-2g^T(K\bar{x}_{pt1})(g(K\bar{x}_{pt1})-LK\bar{x}_{pt1})\end{aligned} \quad (41)$$

Then, by considering (21), (31)-(38) and (41), we can obtain

$$\mathcal{L}V(\bar{x}(t),t)\leq\begin{bmatrix}h(t)\\u(t)\end{bmatrix}^T\Phi(t)\begin{bmatrix}h(t)\\u(t)\end{bmatrix}+q(t) \quad (42)$$

where $\Phi(t)=\begin{bmatrix}\Pi(t) & \bar{\Xi}_{12}(t)+\bar{\Xi}_{13}(t)\bar{E}_v(t)+\bar{\Xi}_{14}(t)\bar{F}_v(t)\\ * & \bar{E}_v^T(t)P_1\bar{E}_v(t)+\bar{F}_v^T(t)P_2\bar{F}_v(t)\end{bmatrix}$ and

$$\begin{aligned}h(t)=[&\bar{x}_m^T(t)\quad \bar{x}_p^T(t)\quad \bar{x}_{mt2}^TK^T\quad \bar{x}_{pt1}^TK^T\\ &g^T(K\bar{x}_{pt1})\quad \bar{x}_m^T(t-h_2)K^T\quad \bar{x}_p^T(t-h_1)K^T\quad \ldots\\ &\bar{x}_m^T(t-rh_2)K^T\quad \bar{x}_p^T(t-rh_1)K^T\quad x_m^T(t)K^T\quad x_p^T(t)K^T\ ]^T\end{aligned}$$

$$\begin{aligned}q(t)=\ &2\sum_{k=1}^{r}\begin{bmatrix}K\bar{x}_m(t-(k-1)h_2)\\K\bar{x}_m(t-kh_2)\end{bmatrix}^T\begin{bmatrix}Z_{km}\\-Z_{km}\end{bmatrix}K\\ &\times\int_{t-kh_2}^{t-(k-1)h_2}\left(\bar{E}(s)K\bar{x}_m(s)+\bar{E}_v(s)u(s)\right)dw_1(s)\\ &+2\sum_{k=1}^{r}\begin{bmatrix}K\bar{x}_p(t-(k-1)h_1)\\K\bar{x}_p(t-kh_1)\end{bmatrix}^T\begin{bmatrix}Z_{kp}\\-Z_{kp}\end{bmatrix}K\\ &\int_{t-kh_1}^{t-(k-1)h_1}\left(\bar{F}(s)K\bar{x}_p(s)+\bar{F}_v(s)u(s)\right)dw_2(s)\end{aligned} \quad (43)$$

$$\Pi(t)=\bar{\Xi}_{11}(t)+\bar{\Xi}_{13}(t)P_1^{-1}\bar{\Xi}_{13}^T(t)+\bar{\Xi}_{14}(t)P_2^{-1}\bar{\Xi}_{14}^T(t)$$

in which $\bar{\Xi}_{11}(t)$ is similar to the matrix Ξ_{11} in (18) with matrices $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_5,\Delta_5,\Delta_6,\Delta_7,\Delta_8$ and $P_1\bar{B},\ P_2\bar{D}$ substituted by matrices $\bar{\Gamma}_1(t),\bar{\Gamma}_2(t),\bar{\Gamma}_3,\bar{\Gamma}_5,\bar{\Delta}_5(t),\bar{\Delta}_6(t),\bar{\Delta}_7(t),\bar{\Delta}_8(t)$, and $P_1\bar{B}(t),\ P_2\bar{D}(t)$, respectively; and

$$\begin{aligned}&\bar{\Xi}_{12}(t)=\left[\bar{A}_{v1}^T(t)P\quad \bar{A}_{v2}^T(t)P\quad 0\quad \ldots\quad 0\quad \bar{A}_{v1}^T(t)K^TS_1\quad \bar{A}_{v2}^T(t)K^TS_2\right]^T\\ &\bar{\Xi}_{13}(t)=\left[P_1\bar{E}(t)K\quad 0\quad 0\quad \mathbf{L}\quad 0\quad 0\quad 0\right]^T,\\ &\bar{\Xi}_{14}(t)=\left[0\quad P_2\bar{F}(t)K\quad 0\quad \mathbf{L}\quad 0\quad 0\quad 0\right]^T\\ &\bar{\Gamma}_1(t)=P_1\bar{A}(t)+\bar{A}^T(t)P_1+K^TR_1K+K^TQ_{1m}K-K^TZ_{1m}K,\\ &\bar{\Gamma}_2(t)=P_2\bar{C}(t)+\bar{C}^T(t)P_2+K^TR_2K+K^TQ_{1p}K-K^TZ_{1p}K,\\ &\bar{\Gamma}_3=(d_2-1)R_1,\bar{\Gamma}_5=-2I\\ &\bar{\Delta}_5(t)=\bar{A}^T(t)K^TS_1,\ \bar{\Delta}_6(t)=\bar{C}^T(t)K^TS_2,\\ &\bar{\Delta}_7(t)=\bar{D}^T(t)K^TS_2,\ \bar{\Delta}_8(t)=\bar{B}^T(t)K^TS_1\end{aligned} \quad (44)$$

If $u(t)=0$, then (42) can be simplified as $\mathcal{L}V(\bar{x}(t),t)\leq h(t)^T\Pi(t)h(t)+q(t)$ and since $\mathbf{E}\{q(t)\}=0$, we have

$$\mathbf{E}\{\mathcal{L}V(\bar{x}(t),t)\}\leq\mathbf{E}\{h(t)^T\Pi(t)h(t)\} \quad (45)$$

Therefore, if $\Pi(t)<0$, based on the Lyapunov stability theory, the filtering error dynamics (12) with $u(t)=0$ is robustly asymptotically mean square stable. Consider that applying Schur complement formula to (16), for $u(t)=0$, yields

$$\Psi_0+\sum_{i=1}^{6}\left(e_i^{-1}\tilde{G}_i\tilde{G}_i^T+e_i\tilde{H}_i^T\tilde{H}_i\right)<0 \quad (46)$$

where $\Psi_0=\begin{bmatrix}\tilde{\Xi}_{11} & \Xi_{13} & \Xi_{14}\\ * & -P_1 & 0\\ * & * & -P_2\end{bmatrix}$, in which $\tilde{\Xi}_{11}$ is similar to the matrix Ξ_{11} in (18) with matrices $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_5 substituted by matrices $\tilde{\Gamma}_1,\tilde{\Gamma}_2,\tilde{\Gamma}_3$ and $\tilde{\Gamma}_5$, respectively; and

$$\begin{aligned}
&\tilde{\Gamma}_1 = P_1\bar{A} + \bar{A}^T P_1 + K^T R_1 K + K^T Q_{1m} K - K^T Z_{1m} K,\\
&\tilde{\Gamma}_2 = P_2\bar{C} + \bar{C}^T P_2 + K^T R_2 K + K^T Q_{1p} K - K^T Z_{1p} K,\\
&\tilde{\Gamma}_3 = (d_2-1)R_1, \quad \tilde{\Gamma}_5 = -2I\\
&\tilde{G}_1 = \left[\bar{G}_1^T P_1 \;\; 0 \;\; \cdots \;\; 0 \;\; \bar{G}_1^T K^T S_1 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T,\\
&\tilde{H}_1 = \left[\bar{H}_1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0\right]\\
&\tilde{G}_2 = \left[\bar{G}_2^T P_1 \;\; 0 \;\; \cdots \;\; 0 \;\; \bar{G}_2^T K^T S_1 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T,\\
&\tilde{H}_2 = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; H_2 \;\; 0 \;\; \cdots \;\; 0\right]\\
&\tilde{G}_3 = \left[0 \;\; 0 \;\; \cdots \;\; 0 \;\; \bar{G}_3^T P_1 \;\; 0\right]^T,\\
&\tilde{H}_3 = \left[H_4 K \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; H_5 \;\; 0 \;\; 0\right]\\
&\tilde{G}_4 = \left[0 \;\; \bar{G}_4^T P_2 \;\; 0 \;\; \cdots \;\; 0 \;\; \bar{G}_4^T K^T S_2 \;\; 0 \;\; 0 \;\; 0\right]^T,\\
&\tilde{H}_4 = \left[0 \;\; \bar{H}_1 \;\; 0 \;\; \cdots \;\; 0\right]\\
&\tilde{G}_5 = \left[0 \;\; \bar{G}_5^T P_2 \;\; 0 \;\; \cdots \;\; 0 \;\; \bar{G}_5^T K^T S_2 \;\; 0 \;\; 0 \;\; 0\right]^T,\\
&\tilde{H}_5 = \left[0 \;\; 0 \;\; H_2 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0\right]\\
&\tilde{G}_6 = \left[0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; \bar{G}_6^T P_2\right]^T,\\
&\tilde{H}_6 = \left[0 \;\; H_4 K \;\; 0 \;\; \cdots \;\; 0 \;\; H_5 \;\; 0 \;\; 0\right]
\end{aligned} \tag{47}$$

Applying Lemma 1 to (46), we have

$$\Psi_0 + \sum_{i=1}^{6}\left(\tilde{G}_i \Delta(t) \tilde{H}_i + \tilde{H}_i^T \Delta^T(t) \tilde{G}_i^T\right) < 0 \tag{48}$$

According to Schur complement formula, (48) is equivalent to $\Pi(t) < 0$ and therefore $\mathbf{E}\{\mathcal{L}V(\bar{x}(t),t)\} < 0$. Thus the filtering error dynamics (12) with $u(t) = 0$ is robustly asymptotically mean square stable. Next, we will establish the $L_2 - L_\infty$ performance for all nonzero $u(t)$. We define

$$J_0 := \mathbf{E}\{V(\bar{x}(t),t)\} - \int_0^t u^T(s)u(s)\,ds \tag{49}$$

Under zero initial condition, by considering $\mathbf{E}\{dw_i(t)\} = 0, i = 1,2$ and (29) it can be seen that

$$\mathbf{E}\{V(\bar{x}(t),t)\} = \mathbf{E}\left\{\int_0^t dV(\bar{x}(s),s)\right\} = \mathbf{E}\left\{\int_0^t \mathcal{L}V(\bar{x}(s),s)\,ds\right\} \tag{50}$$

Since $\mathbf{E}\{q(t)\} = 0$, using (49), (50) and (42) for all nonzero $u(t)$ we have

$$\begin{aligned}
J_0 &= \mathbf{E}\left\{\int_0^t \left[\mathcal{L}V(\bar{x}(s),s) - u^T(s)u(s)\right] ds\right\}\\
&\leq \mathbf{E}\left\{\int_0^t \begin{bmatrix} h(s) \\ u(s)\end{bmatrix}^T \Phi_v(s) \begin{bmatrix} h(s) \\ u(s)\end{bmatrix} ds\right\}
\end{aligned} \tag{51}$$

where $\Phi_v(t) = \begin{bmatrix} \Pi(t) & \Xi_{12}(t) + \Xi_{13}(t)\bar{E}_v(t) + \Xi_{14}(t)\bar{F}_v(t) \\ * & -I + \bar{E}_v^T(t)P_1\bar{E}_v(t) + \bar{F}_v^T(t)P_2\bar{F}_v(t)\end{bmatrix}$.

Consider that by Schur complement formula, it can be seen that for all nonzero $u(t)$, (16) is equivalent to

$$\Psi_{v0} + \sum_{i=1}^{7}\left(\varepsilon_i^{-1}\tilde{G}_i\tilde{G}_i^T + \varepsilon_i\tilde{H}_i^T\tilde{H}_i\right) < 0 \tag{52}$$

where

$$\begin{aligned}
&\Psi_{v0} = \begin{bmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \Xi_{13} & \Xi_{14} \\ * & -I & 0 & 0 \\ * & * & -P_1 & 0 \\ * & * & * & -P_2 \end{bmatrix},\\
&\tilde{\Xi}_{12} = \left[\bar{A}_{v1}^T P_1 \;\; \bar{A}_{v2}^T P_2 \;\; 0 \;\; \cdots \;\; 0 \;\; A_{v1}^T S_1 \;\; A_{v2}^T S_2\right]^T\\
&\tilde{G}_7 = \left[\bar{G}_3^T P_1 \;\; \bar{G}_6^T P_2 \;\; 0 \;\; \cdots \;\; 0 \;\; \bar{G}_3^T K^T S_1 \;\; \bar{G}_6^T K^T S_2 \;\; 0 \;\; 0 \;\; 0\right]^T\\
&\tilde{H}_7 = \left[0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; H_3 \;\; 0 \;\; 0\right]
\end{aligned} \tag{53}$$

Applying Lemma 1 to (52) yields

$$\Psi_{v0} + \sum_{i=1}^{7}\left(\tilde{G}_i \Delta(t) \tilde{H}_i + \tilde{H}_i^T \Delta^T(t) \tilde{G}_i^T\right) < 0 \tag{54}$$

According to Schur complement formula, (54) is equivalent to $\Phi_v(t) < 0$, which ensures $J_0 < 0$, and

$$\begin{aligned}
&\mathbf{E}\left\{\bar{x}_m^T(t)P_1\bar{x}_m(t)\right\} \leq \mathbf{E}\{V(\bar{x}(t),t)\} < \int_0^t u^T(s)u(s)\,ds\\
&\mathbf{E}\left\{\bar{x}_p^T(t)P_2\bar{x}_p(t)\right\} \leq \mathbf{E}\{V(\bar{x}(t),t)\} < \int_0^t u^T(s)u(s)\,ds
\end{aligned} \tag{55}$$

By Schur complement formula, it follows from (17) that

$$\bar{H}_m^T\bar{H}_m < \gamma^2 P_1, \; \bar{H}_p^T\bar{H}_p < \gamma^2 P_2 \tag{56}$$

Hence, from (55) and (56), for all nonzero $u(t)$, we can obtain

$$\begin{aligned}
\mathbf{E}\left\{|e_m(t)|^2\right\} &= \mathbf{E}\left\{e_m^T(t)e_m(t)\right\} = \mathbf{E}\left\{\bar{x}_m^T(t)\bar{H}_m^T\bar{H}_m\bar{x}_m(t)\right\}\\
&< \gamma^2\mathbf{E}\left\{\bar{x}_m^T(t)P_1\bar{x}_m(t)\right\}\\
&< \gamma^2\int_0^t u^T(s)u(s)\,ds \leq \gamma^2\int_0^\infty u^T(s)u(s)\,ds = \gamma^2\|u(t)\|_2^2
\end{aligned} \tag{57}$$

$$\begin{aligned}
\mathbf{E}\left\{|e_p(t)|^2\right\} &= \mathbf{E}\left\{e_p^T(t)e_p(t)\right\} = \mathbf{E}\left\{\bar{x}_p^T(t)\bar{H}_p^T\bar{H}_p\bar{x}_p(t)\right\}\\
&< \gamma^2\mathbf{E}\left\{\bar{x}_p^T(t)P_2\bar{x}_p(t)\right\}\\
&< \gamma^2\int_0^t u^T(s)u(s)\,ds \leq \gamma^2\int_0^\infty u^T(s)u(s)\,ds = \gamma^2\|u(t)\|_2^2
\end{aligned} \tag{58}$$

which implies $\|e_m(t)\|_{E_\infty} < \gamma\,\|u(t)\|_2$ and $\|e_p(t)\|_{E_\infty} < \gamma\,\|u(t)\|_2$, for all nonzero $u(t)$. Thus, the filtering error dynamics (12) is robustly asymptotically mean square stable with the prescribed $L_2 - L_\infty$ attenuation level g. Now we will focus on the design of robust $L_2 - L_\infty$ filter in (10) based on theorem 1 such that, the filtering error dynamics is robustly asymptotically mean square stable with the prescribed $L_2 - L_\infty$ attenuation level g.

Theorem 2: Given an integer $r \ge 1$ and a scalar $g > 0$, if there exist scalars $e_j > 0, j = 1,...,7,$ matrices $V_i > 0 \in \mathbf{R}^{n\times n}$, $W_i > 0 \in \mathbf{R}^{n\times n}, i = 1,2$ and $n\times n$ matrices $R_i \ge 0, i = 1,2$; $Q_{km} \ge 0$; $Q_{kp} \ge 0$; $Z_{km} \ge 0$; $Z_{kp} \ge 0, k = 1,2,...,r$; N_1; N_2; S_1; S_2; $B_h \in \mathbf{R}^{n\times r}$; $D_h \in \mathbf{R}^{n\times r}$ and $M_i \in \mathbf{R}^{n\times q}, i = 1,2$ such that the following ((59)-to(61)) LMIs hold, Then the filtering error dynamics (12) is robustly asymptotically mean square stable with the $L_2 - L_\infty$ attenuation level g:

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} & \Lambda_{16} & \Lambda_{17} & \Lambda_{18} & \Lambda_{19} & \Lambda_{110} & \Lambda_{111} & \Lambda_{112} & \Lambda_{113} \\ * & \Xi_{22} & \Lambda_{23} & \Lambda_{24} & \Lambda_{25} & \Lambda_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -W_1 & -W_1 & 0 & 0 & 0 & 0 & \Pi_1 & 0 & 0 & 0 & 0 \\ * & * & * & -V_1 & 0 & 0 & 0 & 0 & \Pi_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -W_2 & -W_2 & 0 & 0 & 0 & 0 & 0 & \Pi_3 & 0 \\ * & * & * & * & * & -V_2 & 0 & 0 & 0 & 0 & 0 & \Pi_4 & 0 \\ * & * & * & * & * & * & -e_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -e_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -e_3 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -e_4 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -e_5 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -e_6 I & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & -e_7 I \end{bmatrix} < 0 \tag{59}$$

$$\Sigma_1 = \begin{bmatrix} W_1 & W_1 & H_m^T - M_1 \\ * & V_1 & H_m^T \\ * & * & g^2 I \end{bmatrix} > 0, \quad \Sigma_2 = \begin{bmatrix} W_2 & W_2 & H_p^T - M_2 \\ * & V_2 & H_p^T \\ * & * & g^2 I \end{bmatrix} > 0 \tag{60}$$

where

$$\Lambda_{11}=\begin{bmatrix}
\Theta_1 & 0 & 0 & 0 & \tilde{B} & \tilde{Z}_{1m} & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{A}_1 S_1 & 0\\
* & \Theta_2 & \tilde{D} & 0 & 0 & 0 & \tilde{Z}_{1p} & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{A}_2 S_2\\
* & * & \Gamma_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & D^T S_2\\
* & * & * & \Gamma_4 & L & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
* & * & * & * & \Gamma_5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & B^T S_1 & 0\\
* & * & * & * & * & \Gamma_{2m} & 0 & Z_{2m} & 0 & \cdots & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & \Gamma_{2p} & 0 & Z_{2p} & \ddots & \vdots & 0 & 0 & 0\\
* & * & * & * & * & * & * & \ddots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \Gamma_{rm} & 0 & Z_{rm} & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & \Gamma_{rp} & 0 & Z_{rp} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & \Gamma_{(r+1)m} & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & \Gamma_{(r+1)p} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & \Delta_3 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & * & \Delta_4
\end{bmatrix} \tag{61}$$

$$\begin{aligned}
&\Lambda_{12}=\begin{bmatrix}\tilde{A}_{v1}^T & \tilde{A}_{v2}^T & 0 & \cdots & 0 & A_{v1}^T S_1 & A_{v2}^T S_2\end{bmatrix}^T\\
&\Lambda_{13}=\begin{bmatrix}\tilde{E}_1 & 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}^T,\\
&\Lambda_{15}=\begin{bmatrix}0 & \tilde{F}_1 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}^T\\
&\Lambda_{14}=\begin{bmatrix}\tilde{E}_2 & 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}^T,\\
&\Lambda_{16}=\begin{bmatrix}0 & \tilde{F}_2 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}^T\\
&\Lambda_{17}=\begin{bmatrix}-\Pi_1^T & \Pi_5^T & 0 & \cdots & 0 & -\Pi_6^T & 0\end{bmatrix}^T,\\
&\Lambda_{18}=\begin{bmatrix}\Pi_1^T & \tilde{G}_1^T & 0 & \cdots & 0 & \Pi_6^T & 0\end{bmatrix}^T\\
&\Lambda_{110}=\begin{bmatrix}0 & 0 & -\Pi_3^T & \Pi_7^T & 0 & \cdots & 0 & -\Pi_8^T\end{bmatrix}^T,\\
&\Lambda_{19}=\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}^T\\
&\Lambda_{111}=\begin{bmatrix}0 & 0 & \Pi_3^T & \tilde{G}_2^T & 0 & \cdots & 0 & \Pi_8^T\end{bmatrix}^T,\\
&\Lambda_{112}=\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}^T\\
&\Lambda_{113}=\begin{bmatrix}\Pi_1^T & \Pi_2^T & \Pi_3^T & \Pi_4^T & 0 & \cdots & 0 & \Pi_6^T & \Pi_8^T\end{bmatrix}^T\\
&\Lambda_{23}=E_{v1}^T W_1,\ \Lambda_{24}=E_{v1}^T V_1+E_{v2}^T B_h^T,\\
&\Lambda_{25}=F_{v1}^T W_2,\ \Lambda_{26}=F_{v1}^T V_2+F_{v2}^T D_h^T
\end{aligned} \tag{62}$$

in which

$$\tilde{A}_{v1}=\begin{bmatrix}W_1 A_{v1}+e_3 H_4^T H_5\\ V_1 A_{v1}+B_h C_{v1}+e_3 H_4^T H_5\end{bmatrix},\ \tilde{A}_{v2}=\begin{bmatrix}W_2 A_{v2}+e_6 H_4^T H_5\\ V_2 A_{v2}+D_h C_{v2}+e_6 H_4^T H_5\end{bmatrix}$$

$$\tilde{E}_1=\begin{bmatrix}W_1 E_1 & W_1 E_1\end{bmatrix},\ \tilde{E}_2=\begin{bmatrix}V_1 E_1+B_h E_2 & V_1 E_1+B_h E_2\end{bmatrix}$$

$$\tilde{F}_1=\begin{bmatrix}W_2 F_1 & W_2 F_1\end{bmatrix},\ \tilde{F}_2=\begin{bmatrix}V_2 F_1+D_h F_2 & V_2 F_1+D_h F_2\end{bmatrix}$$

$$\begin{aligned}
&\Pi_1=W_1 G_1,\ \Pi_2=\tilde{G}_1+B_h G_3,\ \Pi_5=-\tilde{G}_1+B_h G_3,\\
&\Pi_3=W_2 G_2,\ \Pi_4=\tilde{G}_2+D_h G_4,\ \Pi_7=-\tilde{G}_2+D_h G_4,\\
&\Pi_6=S_1^T G_1,\ \tilde{G}_1=V_1 G_1,\ \Pi_8=S_2^T G_2,\ \tilde{G}_2=V_2 G_2\\
&\Theta_1=\hat{A}_1+\hat{A}_1^T+\begin{bmatrix}\tilde{Q}_m & \tilde{Q}_m\\ \tilde{Q}_m & \tilde{Q}_m\end{bmatrix}+\begin{bmatrix}\tilde{H}_m & \tilde{H}_m\\ \tilde{H}_m & \tilde{H}_m\end{bmatrix},\\
&\hat{A}_1=\begin{bmatrix}-W_1 A_1 & -W_1 A_1\\ -V_1 A_1+B_h C_1+N_1 & -V_1 A_1+B_h C_1\end{bmatrix}\\
&\tilde{Q}_m=Q_{1m}-Z_{1m}+R_1,\ \tilde{H}_m=e_1 H_1^T H_1+e_3 H_4^T H_4\\
&\Theta_2=\hat{A}_2+\hat{A}_2^T+\begin{bmatrix}\tilde{Q}_p & \tilde{Q}_p\\ \tilde{Q}_p & \tilde{Q}_p\end{bmatrix}+\begin{bmatrix}\tilde{H}_p & \tilde{H}_p\\ \tilde{H}_p & \tilde{H}_p\end{bmatrix},\\
&\hat{A}_2=\begin{bmatrix}-W_2 A_2 & -W_2 A_2\\ -V_2 A_2+D_h C_2+N_2 & -V_2 A_2+D_h C_2\end{bmatrix}\\
&\tilde{Q}_p=Q_{1p}-Z_{1p}+R_2,\ \tilde{H}_p=e_4 H_1^T H_1+e_6 H_4^T H_4\\
&\tilde{B}=\begin{bmatrix}W_1 B\\ V_1 B\end{bmatrix},\ \tilde{D}=\begin{bmatrix}W_2 D\\ V_2 D\end{bmatrix},\ \tilde{Z}_{1m}=\begin{bmatrix}Z_{1m}\\ Z_{1m}\end{bmatrix},\\
&\tilde{Z}_{1p}=\begin{bmatrix}Z_{1p}\\ Z_{1p}\end{bmatrix},\ \tilde{A}_1=\begin{bmatrix}-A_1^T\\ -A_1^T\end{bmatrix},\ \tilde{A}_2=\begin{bmatrix}-A_2^T\\ -A_2^T\end{bmatrix}
\end{aligned} \tag{63}$$

In this case, the parameters of the desired filter (10) are given as follows:

$$\begin{aligned}
&A_f=X_1^{-1}N_1 W_1^{-1}Y_1^{-T},\ B_f=X_1^{-1}B_h,\ H_{mf}=\left(Y_1^{-1}W_1^{-1}M_1\right)^T\\
&C_f=X_2^{-1}N_2 W_2^{-1}Y_2^{-T},\ D_f=X_2^{-1}D_h,\ H_{pf}=\left(Y_2^{-1}W_2^{-1}M_2\right)^T
\end{aligned} \tag{64}$$

where X_1, X_2, Y_1 and Y_2 are any nonsingular matrices satisfying

$$X_1 Y_1^T=-V_1 W_1^{-1}+I,\quad X_2 Y_2^T=-V_2 W_2^{-1}+I \tag{65}$$

Proof: From (60), it is clear that $V_i W_i^{-1}-I, i=1,2$ are invertible. Therefore, there are nonsingular matrices X_i and $Y_i, i=1,2$ satisfying (65). Now define nonsingular matrices $\Upsilon_i = \begin{bmatrix} U_i & I \\ Y_i^T & 0 \end{bmatrix}, i=1,2$ with $U_i = W_i^{-1}$ and $P_i = \begin{bmatrix} V_i & X_i \\ X_i^T & T_i \end{bmatrix} > 0, i=1,2$ with $T_i = Y_i^{-1} U_i (V_i - U_i^{-1}) U_i Y_i^{-T}$. Now we set

$$\begin{aligned}
&A_h = Y_1 A_f^T X_1^T \ , \quad B_h = X_1 B_f \ , \\
&U_1^{-1} A_h = N_1^T \ , \quad W_1 = U_1^{-1} \\
&C_h = Y_2 C_f^T X_2^T \ , \quad D_h = X_2 D_f \ , \\
&U_2^{-1} C_h = N_2^T \ , \quad W_2 = U_2^{-1} \\
&Y_1 H_{mf}^T = H_{mh}^T \ , \quad W_1 H_{mh}^T = M_1 \ , \\
&Y_2 H_{pf}^T = H_{ph}^T \ , \quad W_2 H_{ph}^T = M_2 \\
&\mathrm{O} = diag\{\Upsilon_1, \Upsilon_2, I, ..., I, \Upsilon_1, \Upsilon_2, diag_7\{I\}\} \\
&\mathrm{O}_1 = diag\{U_1^{-1}, I, U_2^{-1}, I, I, ..., I, U_1^{-1}, I, U_2^{-1}, I, diag_7\{I\}\} \\
&\mathrm{O}_m = diag\{\Upsilon_1, I\}, \ \mathrm{O}_{1m} = diag\{U_1^{-1}, I, I\}, \\
&\mathrm{O}_p = diag\{\Upsilon_2, I\}, \ \mathrm{O}_{1p} = diag\{U_2^{-1}, I, I\}
\end{aligned} \tag{66}$$

By some algebraic matrix manipulations, we can prove that (59) and (60) are equivalent to

$$\begin{aligned}
&\mathrm{O}_1^T \mathrm{O}^T \Xi \mathrm{O} \mathrm{O}_1 = \Lambda < 0 \\
&\mathrm{O}_{1m}^T \mathrm{O}_m^T \Omega_1 \mathrm{O}_m \mathrm{O}_{1m} = \Sigma_1 > 0 \ , \ \mathrm{O}_{1p}^T \mathrm{O}_p^T \Omega_2 \mathrm{O}_p \mathrm{O}_{1p} = \Sigma_2 > 0
\end{aligned} \tag{67}$$

So, by applying congruence transformations to (59) and (60), we can obtain (16) and (17). Therefore, we can conclude from theorem 1 that the filter (10) with $A_f, B_f, C_f, D_f, H_{mf}$ and H_{pf} defined in (64) ensures the filtering error dynamics (12) to be robustly asymptotically mean square stable with the prescribed $L_2 - L_\infty$ attenuation level g **.**

4. Simulation results

In this section, a simulation study is given to show the effectiveness of the filter design method presented in previous section. Consider the GRN model (6) with three nodes and following numerical values. This example is an extension to one in [12].

$$A_1 = diag\{3,3,3\}, \ C_1 = diag\{0.72, 0.1, 1.3\},$$

$$B = \begin{bmatrix} 0 & 0 & -1.2 \\ -0.02 & 0 & 0 \\ 0 & -0.2 & 0 \end{bmatrix}$$

$$A_2 = diag\{3.5, 3.5, 3.5\}, \ C_2 = diag\{1.2, 3.1, 0.03\},$$

$$D = diag\{1.21, 2.12, 0.22\}$$

$$E_1 = diag\{0.53, 0.41, 0.42\}, \ F_1 = diag\{0.51, 0.41, 0.42\}$$

The nonlinear function is $g_i(x_{pi}) = x_{pi}^2 / (1 + x_{pi}^2), \ i = 1,2,3$, i.e. $L = diag\{0.6, 0.6, 0.6\}$. The time-varying delays and disturbance are taken from [21] as:

$t_1(t) = t_2(t) = 1.41 + 0.09\sin(t)$, $u(t) = [1\ 5\ 5]^T \sin(t)\exp(-2t)$

and additional matrices are chosen as follows:

$$E_2 = F_2 = diag\{0.6, 0.5, 0.5\}, \ H_m = H_p = diag\{-0.1, -0.6, 0.5\}$$

$$A_{v1} = E_{v1} = diag\{-0.8, -0.2, -0.2\}, \ A_{v2} = F_{v1} = diag\{-0.4, 0.3, -0.15\}$$

$$C_{v1} = E_{v2} = diag\{0.2, -0.4, 0.2\}, \quad C_{v2} = F_{v2} = diag\{-0.2, -0.7, 0.6\}$$

$$H_1 = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0 & 0.2 & 0 \\ 0 & -0.1 & 0.2 \end{bmatrix}, \ H_2 = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.1 & 0.1 & 0 \\ 0.1 & 0.5 & 0.1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.1 & 0 & -0.1 \\ 0.1 & 0.1 & 0.5 \\ 0.1 & 0 & 0 \end{bmatrix}, \ G_2 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & -0.1 \\ -0.1 & -0.05 & 0 \end{bmatrix}$$

$$H_3 = diag\{0.2, 0.2, 0.9\}, \ H_4 = diag\{0.3, 0.1, 0.8\}, \ H_5 = diag\{0.5, 0.9, 0.1\}$$

$$G_3 = diag\{0.4, 0.3, 0.4\}, \ G_4 = diag\{0.2, 0.35, 0.25\}$$

The unknown time-varying matrix $\Delta(t)$ satisfying condition (9) is taken as $\Delta(t) = diag\{\sin(t), \cos(t), -\sin(t)\}$. For $g = 1.2$, $r = 1$ and by using the Matlab LMI control Toolbox, the LMIs in (59) and (60) are solved and the filter parameters in (10) are obtained as:

$$A_f = \begin{bmatrix} -4.2976 & 0.0741 & -0.0338 \\ 1.2546 & -3.2182 & 0.1424 \\ -0.0966 & -0.0906 & -3.9441 \end{bmatrix},$$

$$B_f = \begin{bmatrix} -0.1168 & -0.0406 & -0.0517 \\ -0.0406 & -0.7052 & -0.1736 \\ -0.0517 & -0.1736 & 0.4832 \end{bmatrix}$$

$$C_f = \begin{bmatrix} -4.1322 & -0.2245 & -0.0328 \\ -0.0559 & -1.6643 & -0.2849 \\ -0.3147 & 7.8405 & -5.7383 \end{bmatrix},$$

$$D_f = \begin{bmatrix} -2.0163 & -0.0873 & 0.1524 \\ -0.0873 & 0.1760 & 0.2721 \\ 0.1524 & 0.2721 & 1.9167 \end{bmatrix}$$

$$H_{mf} = \begin{bmatrix} 0.2358 & -0.0036 & 0.0063 \\ -0.1252 & 0.3742 & 0.0194 \\ -0.0255 & -0.0145 & -0.0818 \end{bmatrix},$$

$$H_{pf} = \begin{bmatrix} 0.0486 & -0.0119 & 0.0039 \\ -0.0170 & 0.6384 & -0.0917 \\ 0.0011 & 0.1146 & -0.0553 \end{bmatrix}$$

Figures (1)-(6) give the time histories of the states of the original GRN (6) and their estimates. As seen, the expected robustly asymptotically mean square stability has been guaranteed and the estimates converge to their true values in the presence of stochastic noises, allowable nonlinearities, norm-bounded uncertainties and time-varying delays.

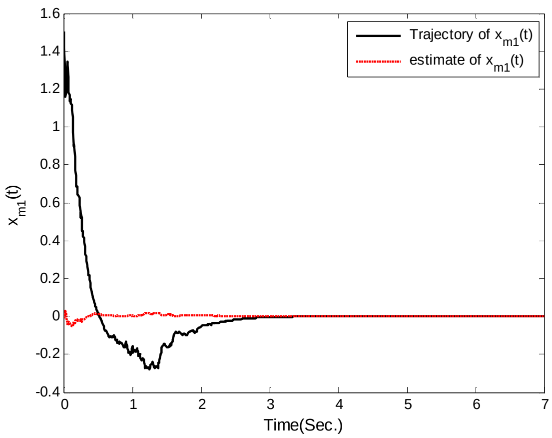

Fig. 1. Trajectory and estimate of $x_{m_1}(t)$

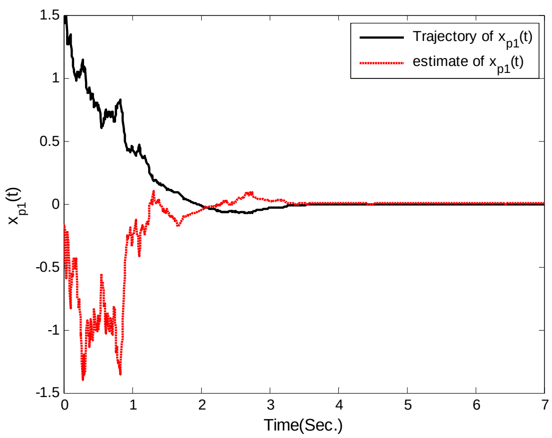

Fig. 2. Trajectory and estimate of $x_{p_1}(t)$

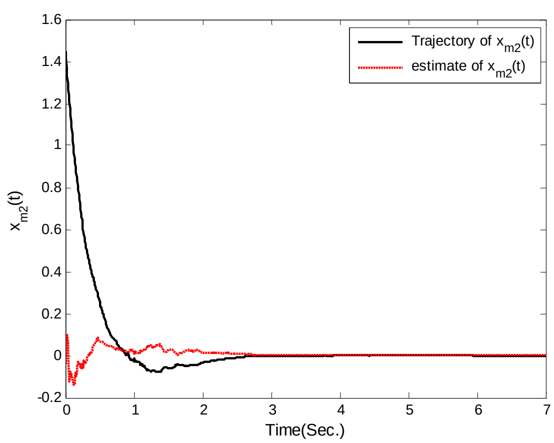

Fig. 3. Trajectory and estimate of $x_{m_2}(t)$

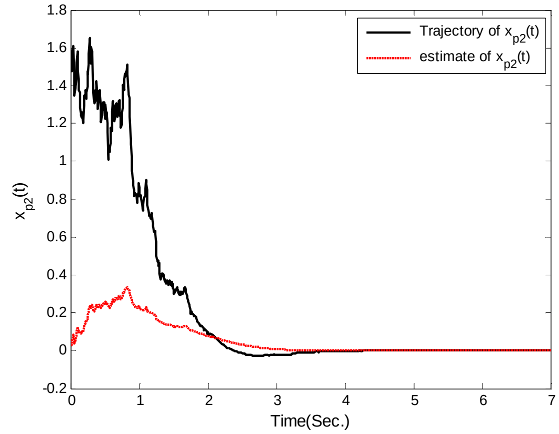

Fig. 4. Trajectory and estimate of $x_{p_2}(t)$

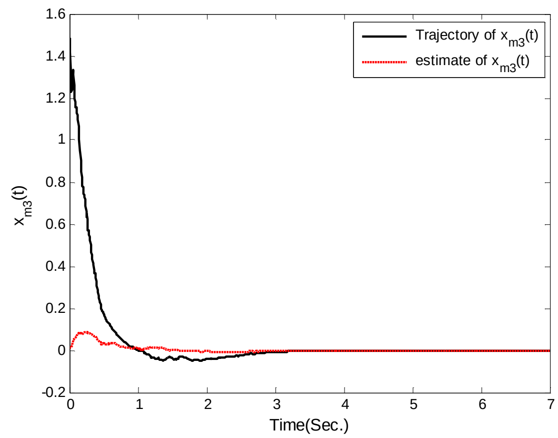

Fig. 5. Trajectory and estimate of $x_{m_3}(t)$

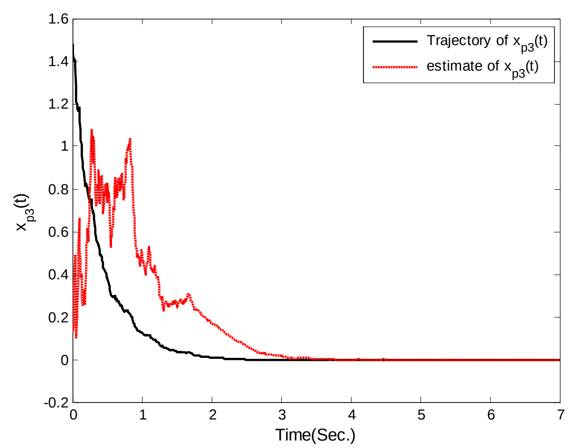

Fig. 6. Trajectory and estimate of $x_{p_3}(t)$

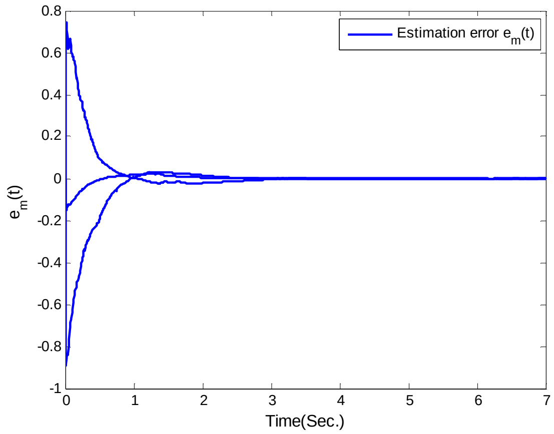

Fig. 7. Estimation error of $e_m(t)=z_m(t)-\hat{z}_m(t)$

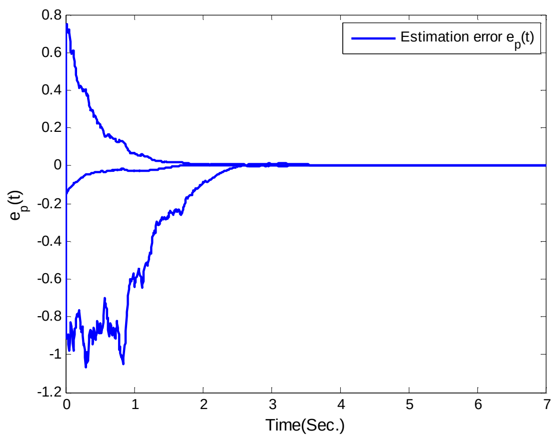

Fig. 8. Estimation error of $e_p(t)=z_p(t)-\hat{z}_p(t)$

Figures (7) and (8) show that the estimation error converges to zero asymptotically. This convergence provides stability condition in definition 2.

More importantly, under zero initial condition and presence of exogenous input, we can obtain L_2-L_∞ attenuation levels:

$$\frac{\|e_m(t)\|_{E_\infty}}{\|u(t)\|_2}=1.0361 \text{ and } \frac{\|e_p(t)\|_{E_\infty}}{\|u(t)\|_2}=1.0815$$

both less than the prescribed level $g=1.2$.

5. CONCLUSIONS

This paper has been addressed a delay-dependent robust L_2-L_∞ filter design method for genetic regulatory networks. At first, a nonlinear uncertain GRN model has been introduced such that the parameter uncertainties (time-varying and norm-bounded) are considered, delays are time-varying, stochastic noises appear at both the state and measurement equations and this model is considered under stochastic noises and disturbance simultaneously. Thus, the model provided is more realistic than previous proposed models. Based on the Lyapunov-Krasovskii functional method, delay decomposition approach, the stochastic integral inequality and free weighting matrix technique, delay-dependent sufficient conditions have been derived in the form of LMIs, which ensure robust asymptotical stability of the filtering error dynamics with the prescribed L_2-L_∞ attenuation level. Since the results are delay-dependent, they have less conservativeness than previous results. Then the filter parameters have been determined in terms of the solution of LMIs. Finally, a simulation example has been given which demonstrates the states of estimator converge to their true values in the presence of stochastic noises and disturbance, allowable nonlinearities, parameter uncertainties and time-varying delays.

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