Equations of Motion Extraction for a Three Axes Gimbal System

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Abstract
Inertial stabilization of a sensor mounted on non-stationary platforms is an important task in many applications such as image processing, astronomical telescopes, and tracking systems. For this purpose, some type of gimbaling arrangements is typically used. For implementing the LOS stabilization, usually a two-axes gimbal system is used which the sensor is mounted in the inner gimbal. The dynamic modeling and designing the control system for two axes gimbal system have been studied extensively, but there is a few works on three axes gimbal system. In this paper, the equation of motion for a three axes gimbal system is derived by the moment equation. The effect of angular velocities of the base into the gimbaled dynamic system and cross-coupling between gimbals are presented. In addition, some critical notes are presented for constructing the gimbal assembly. Moreover, a model based control strategy is proposed for controlling the gimbal dynamics.

Keywords: Gimbal system, Inertial Stabilization Platform, Equation of Motion, Dynamic System.

1. Introduction

nertial stabilization of a sensor (such as Television, Telescope, IR, Radar, and Laser) mounted on non-stationary platforms is an important task in many applications such as image processing, guided missiles, astronomical telescopes, and tracking systems [1-5]. In such systems, the Line-of-Sight (LOS) of sensor must be pointed to a constant or a moving target while the base of the sensor is a moving platform. For this purpose, some type of gimbaling arrangements equipped with inertial sensors is typically used [2, 6]. For implementing the LOS stabilization, usually a two-axes gimbal system is used which the sensor is mounted in the inner gimbal. Two rate gyroscopes are placed beside the sensor which measures the angular velocity of the sensor in the inertial space. The output signal of the gyroscopes is used as feedback to the gimbal torquers through a control system to make the LOS of the sensor follow certain rate commands [7-9].

In many applications, LOS stabilization is enough for reaching the requirements [10, 12]. However, in some applications such as camera stabilization, the LOS direction could be stabilized with the help of mentioned gimbal configuration, but the image rotation cannot be compensated and cause performance degradation. For doing this, an extra gimbal should be provided to achieve image rotation compensation and better isolation from the base movements and disturbances.

Different kind of disturbances is affecting the sensor including platform movements, cross coupling between gimbals, and some imperfections in the gimbals mechanism. To keep the sensor direction constant in the inertial space, these disturbances should be rejected or attenuated. Therefore, dynamic modeling of the gimbaled system is necessary for identifying these disturbances and designing a control system structure. The dynamic modeling and designing the control system for two-axes gimbal system have been studied extensively [10, 13-17], but there is a few works on three-axes gimbal system.

In this paper, the equation of motion of a three axes gimbal system is derived by the moment equation. The effect of angular velocities of the base into the gimbaled dynamic system and cross-coupling between gimbals are discussed. In addition, some critical notes are presented for constructing the gimbal assembly. Moreover, a control strategy is presented for controlling the gimbal dynamic.

The organization of this paper is as follows. In Section II, the kinematics of the gimbal system is presented. In Section III, the equation of motion for the system is derived by the moment equation. In Section IV, a control strategy is proposed and the inverse kinematics is presented. Finally, Section V concludes the paper.

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2. Gimbal System Kinematics

Consider a three axes gimbal system and related reference frames as depicted in Figure 1. As it shown in this figure, five reference frames are introduced: the inertial coordinate frame I, a body-fixed frame B, and three frames Y, P, and R fixed to the yaw, pitch, and roll gimbals, respectively. The sensor is placed and fixed on the roll gimbal (frame R) which $\mathbf{x}_R$ -axis coincides with the sensor optical axis or the sensor LOS. Axes directions in frame I and B are as usual as flight dynamics which z-axis pointing down and y-axis is along the right wing [18]. All frames origins are coincident and the gimbals are considered as rigid bodies.

The body-fixed frame B is carried into frame Y by a positive angle of rotation q_Y about the $\mathbf{z}_B$ -axis. Frame Y is carried into frame P by a positive angle of rotation q_P about the $\mathbf{y}_Y$ -axis. Finally, frame P is carried into frame R by a positive angle of rotation q_R about the $\mathbf{x}_P$ -axis. Hence, we have the following transformations with respect to these rotations:

$$\mathbf{C}_B^Y = \begin{bmatrix} c_Y & s_Y & 0 \\ -s_Y & c_Y & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

$$\mathbf{C}_Y^P = \begin{bmatrix} c_P & 0 & -s_P \\ 0 & 1 & 0 \\ s_P & 0 & c_P \end{bmatrix}, \tag{2}$$

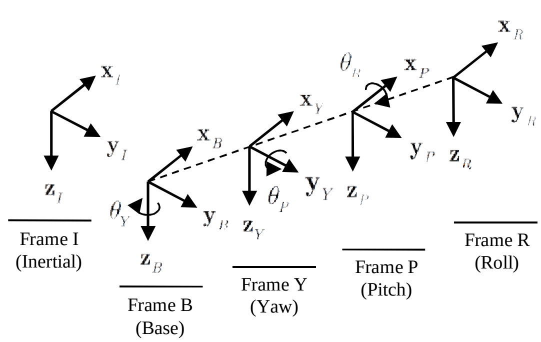

Fig 1: Reference frames of a three-axes gimbal system

$$\mathbf{C}_P^R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_R & s_R \\ 0 & -s_R & c_R \end{bmatrix}, \tag{3}$$

where $\mathbf{C}_i^j$ is the transformation from frame i to j, and $c_i = \cos q_i$, $s_i = \sin q_i$.

Assumption 1: It is assumed that

$$\boldsymbol{\omega}_{IB}^B = \begin{bmatrix} p & q & r \end{bmatrix}^T, \tag{4}$$

is measured, where $\boldsymbol{\omega}_{IB}^B$ is the angular velocity of frame B respect to frame I, and introduced in frame B. Note that most of moving platforms which the sensor mounted on them (such as helicopters, missiles, ships, and etc.) uses Inertial Measurement Unit (IMU) for navigation system. Hence, $\boldsymbol{\omega}_{IB}^B$ is measured and available.

Angular velocities of frame Y, P, and R respect to frame I, and introduced in its own frames are:

$$\begin{aligned} \boldsymbol{\omega}_{IY}^Y &= \boldsymbol{\omega}_{IB}^Y + \boldsymbol{\omega}_{BY}^Y = \mathbf{C}_B^Y \boldsymbol{\omega}_{IB}^B + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_Y \end{bmatrix} \\ &= \begin{bmatrix} pc_Y + qs_Y \\ qc_Y - ps_Y \\ r + \dot{q}_Y \end{bmatrix} = \begin{bmatrix} p_Y \\ q_Y \\ r_Y \end{bmatrix}, \end{aligned} \tag{5}$$

$$\begin{aligned} \boldsymbol{\omega}_{IP}^P &= \boldsymbol{\omega}_{IY}^P + \boldsymbol{\omega}_{YP}^P = \mathbf{C}_Y^P \boldsymbol{\omega}_{IY}^Y + \begin{bmatrix} 0 \\ \dot{q}_P \\ 0 \end{bmatrix} = \begin{bmatrix} p_Y c_P - r_Y s_P \\ q_Y + \dot{q}_P \\ p_Y s_P + r_Y c_P \end{bmatrix} \\ &= \begin{bmatrix} c_P\left(pc_Y + qs_Y\right) - s_P\left(r + \dot{q}_Y\right) \\ qc_Y - ps_Y + \dot{q}_P \\ s_P\left(pc_Y + qs_Y\right) + c_P\left(r + \dot{q}_Y\right) \end{bmatrix} = \begin{bmatrix} p_P \\ q_P \\ r_P \end{bmatrix}, \end{aligned} \tag{6}$$

$$\begin{aligned} \boldsymbol{\omega}_{IR}^R &= \boldsymbol{\omega}_{IP}^R + \boldsymbol{\omega}_{PR}^R = \mathbf{C}_P^R \boldsymbol{\omega}_{IP}^P + \begin{bmatrix} \dot{q}_R \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_P + \dot{q}_R \\ q_P c_R + r_P s_R \\ -q_P s_R + r_P c_R \end{bmatrix} = \begin{bmatrix} p_R \\ q_R \\ r_R \end{bmatrix} \\ &= \begin{bmatrix} \dot{q}_R - s_P\left(r + \dot{q}_Y\right) + c_P\left(pc_Y + qs_Y\right) \\ s_R\left(s_P\left(pc_Y + qs_Y\right) + c_P\left(r + \dot{q}_Y\right)\right) + c_R\left(\dot{q}_P + qc_Y - ps_Y\right) \\ c_R\left(s_P\left(pc_Y + qs_Y\right) + c_P\left(r + \dot{q}_Y\right)\right) - s_R\left(\dot{q}_P + qc_Y - ps_Y\right) \end{bmatrix}, \end{aligned} \tag{7}$$

where $\mathbf{C}_B^Y$, $\mathbf{C}_Y^P$, and $\mathbf{C}_P^R$ are introduced in (1)-(3). The inertia matrices of the gimbals are

$$\mathbf{J}_Y = \mathrm{diag}\left\{J_{Yx}, J_{Yy}, J_{Yz}\right\}, \tag{8}$$

$$\mathbf{J}_P = \mathrm{diag}\left\{J_{Px}, J_{Py}, J_{Pz}\right\}, \tag{9}$$

$$\mathbf{J}_R = \mathrm{diag}\left\{J_{Rx}, J_{Ry}, J_{Rz}\right\}, \tag{10}$$

where $\mathrm{diag}\left\{J_{kx}, J_{ky}, J_{kz}\right\}$ refers to a real matrix with diagonal elements J_{kx}, J_{ky}, J_{kz} ($k = Y, P, R$). For simplicity the off-diagonal elements of inertia matrices, the products of inertia, are assumed to be neglected and just the moments of inertia are considered. In addition, it is assumed that the centers of mass of gimbals are located in the centers of their common rotation. In other words, it is assumed that there is no mass unbalance in the gimbals. Furthermore, external lumped torques τ_Y , τ_P , and τ_R about $\mathbf{z}_B$, $\mathbf{y}_Y$,and $\mathbf{x}_P$ axes, respectively, are applied to gimbals from motor and other external disturbance torques.

3. Equations of Motion

Consider a rigid body and a reference frame x, y, and z fixed to this body. Also, assume that this body has inertia matrix $\mathbf{J}$ and rotate in an inertial frame with angular velocity $\boldsymbol{\omega}$, then the angular momentum of the body will be $\mathbf{H}=\mathbf{J}\boldsymbol{\omega}$. Hence, The moment equation for the body is $\boldsymbol{\tau}=d\mathbf{H}/dt+\boldsymbol{\omega}\times\mathbf{H}$, where $\boldsymbol{\tau}$ is the total external torques [19]. In the following, the equation of motion for the gimbals will be derived separately with the help of the moment equation.

2.1. Roll Channel

The angular momentum of the roll gimbal is

$$\mathbf{H}_R=\mathbf{J}_R\boldsymbol{\omega}_{IR}^R=\begin{bmatrix} J_{Rx}p_R \\ J_{Ry}q_R \\ J_{Rz}r_R \end{bmatrix}, \tag{11}$$

where the parameters are introduced in (7) and (10). So, the moment equation for the roll gimbal can be written as

$$\boldsymbol{\tau}_R=\frac{d\mathbf{H}_R}{dt}+\boldsymbol{\omega}_{IR}^R\times\mathbf{H}_R . \tag{12}$$

Since the roll gimbal can only rotate about its x-axis, the moment equation (12) about this axis with respect to (11) and (7) is

$$\tau_R=J_{Rx}\dot{p}_R+\left(J_{Rz}-J_{Ry}\right)q_R r_R . \tag{13}$$

With substituting (7) in (13) and derivation p_R, (13) can be written as

$$\begin{aligned} J_{Rx}\ddot{\theta}_R &= \tau_R+J_{Rx}\left(\dot{\theta}_P c_P\left(r+\dot{\theta}_Y\right)+\dot{\theta}_P s_P\left(pc_Y+qs_Y\right)\right. \\ &\left.+s_p\left(\dot{r}+\ddot{\theta}_Y\right)-c_P\left(\dot{p}c_Y+\dot{q}s_Y-\dot{\theta}_Y ps_Y+\dot{\theta}_Y qc_Y\right)\right)-\left(J_{Rz}-J_{Ry}\right)\times \\ &\left(s_R\left(s_P\left(pc_Y+qs_Y\right)+c_P\left(r+\dot{\theta}_Y\right)\right)+c_R\left(\dot{\theta}_P+qc_Y-ps_Y\right)\right)\times \\ &\left(c_R\left(s_P\left(pc_Y+qs_Y\right)+c_P\left(r+\dot{\theta}_Y\right)\right)-s_R\left(\dot{\theta}_P+qc_Y-ps_Y\right)\right). \end{aligned} \tag{14}$$

Assumption 2: It is assumed that

$$J_{Rz}=J_{Ry} . \tag{15}$$

Remark 1: The assumption 2 is a practical and critical point and should be considered in the mechanism design procedure. Considering this assumption in implementation phase will be reduced interaction, increase the performance, and simplify the controller design.

With considering Assumption 2, (14) can be written as

$$J_{Rx}\ddot{\theta}_R=\tau_R+\tau_{RI}+\tau_{RB} \tag{16}$$

where

$$\tau_{RI}=J_{Rx}\left(\dot{\theta}_Y\dot{\theta}_P c_P+\ddot{\theta}_Y s_p\right) \tag{17}$$

is the interaction between yaw and pitch channels to roll and

$$\begin{aligned} \tau_{RB} &= J_{Rx}\left(\dot{\theta}_P rc_P+\dot{r}s_p+\dot{\theta}_P s_P\left(pc_Y+qs_Y\right)\right. \\ &\left.-c_P\left(\dot{p}c_Y+\dot{q}s_Y-\dot{\theta}_Y ps_Y+\dot{\theta}_Y qc_Y\right)\right) \end{aligned} \tag{18}$$

is the effect of base movements to roll channel.

From (17), it is obvious that in the case $\theta_P=\pm 90^\circ$ the rotation axis of yaw and roll channel will be coincident and have same effect on the sensor. Hence, how much the pitch rotation deviate from zero, the interaction between the yaw and roll channel will be increased. In addition, changing the angular velocity of the base around z-axis $(\dot{r})$ has the same effect on the roll channel. Moreover, in (17), $J_{Rx}\dot{\theta}_Y\dot{\theta}_P c_P$ is the coriolis effect from yaw and pitch channels to the roll channel.

2.2 Pitch Channel

The angular momentum of the pitch gimbal is

$$\begin{aligned} \mathbf{H}_P &= \mathbf{J}_P\boldsymbol{\omega}_{IP}^P+\left(\mathbf{C}_P^R\right)^T\mathbf{J}_R\boldsymbol{\omega}_{IR}^R \\ &= \begin{bmatrix} J_{Px}p_P+J_{Rx}\left(p_P+\dot{\theta}_R\right) \\ J_{Py}q_P+J_{Ry}c_R\left(q_P c_R+r_P s_R\right)-J_{Rz}s_R\left(-q_P s_R+r_P c_R\right) \\ J_{Pz}r_P+J_{Ry}s_R\left(q_P c_R+r_P s_R\right)+J_{Rz}c_R\left(-q_P s_R+r_P c_R\right) \end{bmatrix}. \end{aligned} \tag{19}$$

where the second part of the equation is the angular momentum of the roll gimbal transformed to pitch frame. The moment equation for the roll gimbal is

$$\boldsymbol{\tau}_P=\frac{d\mathbf{H}_P}{dt}+\boldsymbol{\omega}_{IP}^P\times\mathbf{H}_P . \tag{20}$$

Since the pitch gimbal can only rotate about its y-axis, the moment equation (20) about this axis with respect to (19) and (6) is

$$\begin{aligned} \tau_P &= \dot{q}_P\left(J_{Py}+J_{Ry}c_R^2+J_{Rz}s_R^2\right)+\dot{r}_P s_R c_R\left(J_{Ry}-J_{Rz}\right) \\ &+q_p\left(2\dot{\theta}_R s_R c_R+p_P s_R c_R\right)\left(J_{Rz}-J_{Ry}\right) \\ &+r_P\dot{\theta}_R\left(\left(s_R^2-c_R^2\right)\left(J_{Rz}-J_{Ry}\right)+J_{Rx}\right) \\ &+p_P r_P\left(J_{Px}+J_{Rx}-J_{Pz}-J_{Ry}s_R^2-J_{Rz}c_R^2\right). \end{aligned} \tag{21}$$

With substituting (6) in (21) and derivation q_P and r_P, (21) can be written as

$$\begin{aligned}
&J_{PT}\ddot{q}_P = t_P + J_{PT}\left(\dot{p}s_Y - \dot{q}c_Y + \dot{q}_Y q s_Y + \dot{q}_Y p c_Y\right)\\
&+\left(\dot{q}_P c_P\left(pc_Y + qs_Y\right) + s_P\left(\dot{p}c_Y + \dot{q}s_Y - \dot{q}_Y ps_Y + \dot{q}_Y qc_Y\right)\right.\\
&-c_P\left(\dot{r} + \ddot{q}_Y\right) - \dot{q}_P s_P\left(r + \dot{q}_Y\right)\Big)s_R c_R\left(J_{Rz} - J_{Ry}\right)\\
&+\left(qc_Y - ps_Y + \dot{q}_P\right)\left(J_{Ry} - J_{Rz}\right)\Big(2\dot{q}_R s_R c_R +\\
&\left(c_P\left(pc_Y + qs_Y\right) - s_P\left(r + \dot{q}_Y\right)\right)s_R c_R\Big)\\
&-\dot{q}_R\left(s_P\left(pc_Y + qs_Y\right) + c_P\left(r + \dot{q}_Y\right)\right)\left(\left(s_R^2 - c_R^2\right)\left(J_{Rz} - J_{Ry}\right) + J_{Rx}\right)\\
&-\left(c_P\left(pc_Y + qs_Y\right) - s_P\left(r + \dot{q}_Y\right)\right)\left(s_P\left(pc_Y + qs_Y\right) + c_P\left(r + \dot{q}_Y\right)\right)\times\\
&\left(J_{Px} + J_{Rx} - J_{Pz} - J_{Ry}s_R^2 - J_{Rz}c_R^2\right)
\end{aligned}\tag{22}$$

where $J_{PT} = J_{Py} + J_{Ry}c_R^2 + J_{Rz}s_R^2$ is the total moment of inertia of the pitch gimbal bout y-axis.

With considering Assumption 2, (22) can be written as

$$\left(J_{Py} + J_{Ry}\right)\ddot{q}_P = t_P + t_{PI} + t_{PB}, \tag{23}$$

where

$$t_{PI} = -J_{Rx}\dot{q}_Y\dot{q}_R c_P + 2\dot{q}_Y^2 s_P c_P\left(J_{Px} + J_{Rx} - J_{Pz} - J_{Ry}\right) \tag{24}$$

is the interaction between yaw and roll channels to pitch and

$$\begin{aligned}
&t_{PB} = \left(J_{Py} + J_{Ry}\right)\left(\dot{p}s_Y - \dot{q}c_Y + \dot{q}_Y qs_Y + \dot{q}_Y pc_Y\right)\\
&+\dot{q}_Y\left(\left(s_P^2 - c_P^2\right)\left(pc_Y + qs_Y\right) + 2s_P c_P r\right)\left(J_{Px} + J_{Rx} - J_{Pz} - J_{Ry}\right)\\
&-\Big(\left(2c_P\left(pc_Y + qs_Y\right) - 2s_P r - s_P\dot{q}_Y\right)\left(s_P\left(pc_Y + qs_Y\right) + c_P r\right)\\
&+\dot{q}_Y c_P\left(c_P\left(pc_Y + qs_Y\right) - s_P r\right)\Big)\left(J_{Px} + J_{Rx} - J_{Pz} - J_{Ry}\right)\\
&-J_{Rx}\dot{q}_R\left(s_P\left(pc_Y + qs_Y\right) + rc_P\right),
\end{aligned}\tag{25}$$

is the effect of base movements to pitch channel.

Remark 2: If in the mechanism design the relation between moments of inertial be how $\left(J_{Px} + J_{Rx} - J_{Pz} - J_{Ry}\right) \approx 0$, the gimbal interactions and effect of base movements on the pitch channel would be reduced.

2.3. Yaw Channel

The angular momentum of the yaw gimbal is

$$\begin{aligned}
&\mathbf{H}_Y = \mathbf{J}_Y\boldsymbol{\omega}_{IY}^Y + \left(\mathbf{C}_Y^P\right)^T\left(\mathbf{J}_P\boldsymbol{\omega}_{IP}^P + \left(\mathbf{C}_P^R\right)^T\mathbf{J}_R\boldsymbol{\omega}_{IR}^R\right)\\
&=\begin{bmatrix} J_{Yx}p_Y + \dot{q}_R J_{Rx}c_P + p_P\left(J_{Px}c_P + J_{Rx}c_P\right)\\ J_{Yy}q_Y + q_P\left(J_{Py} + J_{Ry}c_R^2 + J_{Rz}s_R^2\right)\\ J_{Yz}r_Y - \dot{q}_R J_{Rx}s_P + p_P\left(-J_{Px}s_P + J_{Rx}s_P\right)\end{bmatrix}\\
&+\begin{bmatrix} r_P s_P\left(J_{Pz} + J_{Ry}s_R^2 + J_{Rz}c_R^2\right) + q_P s_P c_R s_R\left(J_{Ry} - J_{Rz}\right)\\ r_P c_R s_R\left(J_{Ry} - J_{Rz}\right)\\ r_P c_P\left(J_{Pz} + J_{Ry}s_R^2 + J_{Rz}c_R^2\right) + q_P c_P c_R s_R\left(J_{Ry} - J_{Rz}\right)\end{bmatrix}.
\end{aligned}\tag{26}$$

where the parameters are introduced in (2)-(10). In this equation, the angular momentum of the pitch and roll gimbals transformed to yaw frame. The moment equation for the yaw gimbal is

$$\boldsymbol{\tau}_Y = \frac{d\mathbf{H}_Y}{dt} + \boldsymbol{\omega}_{IY}^Y \times \mathbf{H}_Y\ . \tag{27}$$

Since the yaw gimbal can only rotate about its z-axis, the moment equation (27) about this axis is

$$\begin{aligned}
&t_Y = J_{Yx}\dot{r}_Y - J_{Rx}\ddot{q}_R s_R + J_{Yy}q_Y p_Y + \dot{q}_P c_P\left(p_Y c_P - r_Y s_P\right)\times\\
&\left(J_{Rx} - J_{Px}\right) - J_{Rx}\dot{q}_R\dot{q}_P c_P + \left(\dot{p}_Y c_P - \dot{q}_P p_Y s_P - \dot{r}_Y s_P - \dot{q}_P r_Y c_P\right)\times\\
&\left(J_{Rx} - J_{Px}\right)s_P + \left(\dot{p}_Y s_P + \dot{r}_Y p_Y c_P - \dot{q}_P r_Y s_P\right)\times\\
&\left(J_{Pz} + J_{Ry}s_R^2 + J_{Rz}c_R^2\right) - \dot{q}_P s_P\left(p_Y s_P + r_Y c_P\right)\times\\
&\left(J_{Pz} + J_{Ry}s_R^2 + J_{Rz}c_R^2\right) + 2\dot{q}_R s_R c_R c_P\left(p_Y s_P + r_Y c_P\right)\times\\
&\left(J_{Ry} - J_{Rz}\right) - q_Y c_P\left(p_Y c_P - r_Y s_P\right)\left(J_{Px} + J_{Rx}\right)\\
&+c_P c_R s_R\left(\dot{q}_Y + \ddot{q}_P\right)\left(J_{Ry} - J_{Rz}\right) - J_{Yx}p_Y q_Y - \dot{q}_R J_{Rx}c_P q_Y\\
&+\left(q_Y + \dot{q}_P\right)\left(-\dot{q}_P s_P c_R s_R - \dot{q}_R c_P s_R^2 + \dot{q}_R c_P c_R^2\right)\left(J_{Ry} - J_{Rz}\right)\\
&+\left(-q_Y s_P\left(J_{Pz} + J_{Ry}s_R^2 + J_{Rz}c_R^2\right) + p_Y c_R s_R\left(J_{Ry} - J_{Rz}\right)\right)\times\\
&\left(p_Y s_P + r_Y c_P\right) + p_Y\left(q_Y + \dot{q}_P\right)\left(J_{Py} + J_{Ry}c_R^2 + J_{Rz}s_R^2\right)\\
&-q_Y s_P c_R^2\left(q_Y + \dot{q}_P\right)\left(J_{Ry} - J_{Rz}\right).
\end{aligned}\tag{28}$$

With substituting (5) in (28) and derivation p_Y, q_Y, and r_Y, (28) can be written as

$$\begin{aligned}
&J_{YTT}\ddot{q}_Y = t_Y - \dot{r}J_{YTT} - \ddot{q}_P c_P c_R s_R\left(J_{Ry} - J_{Rz}\right)\\
&-\left(\dot{p}c_Y + \dot{q}s_Y - \dot{q}_Y ps_Y + \dot{q}_Y qc_Y\right)\left(c_P s_P\left(J_{Rx} - J_{Px}\right) + s_P J_{PTZ}\right)\\
&-c_P c_R s_R\left(\dot{q}c_Y - \dot{p}s_Y - \dot{q}_Y qs_Y - \dot{q}_Y pc_Y\right)\left(J_{Ry} - J_{Rz}\right)\\
&-\left(pc_Y + qs_Y\right)\Big(\left(\dot{q}_P c_P - \dot{q}_P s_P^2\right)\left(J_{Rx} - J_{Px}\right) + \dot{q}_P s_P^2 J_{PTZ} +\\
&+2\dot{q}_R s_R c_R c_P s_P\left(J_{Ry} - J_{Rz}\right) + \dot{q}_P\left(J_{Py} + J_{Ry}c_R^2 + J_{Rz}s_R^2\right)\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\left(qc_Y - ps_Y\right)\Big(-\dot{q}_R J_{Rx} c_P + \\
&\quad +\left(-\dot{q}_P s_P c_R s_R - \dot{q}_R c_P s_R^2 + \dot{q}_R c_P c_R^2 - \dot{q}_P s_P c_R^2\right)\left(J_{Ry} - J_{Rz}\right)\Big) \\
&-\left(r + \dot{q}_Y\right)\begin{pmatrix} -2\dot{q}_P c_P s_P \left(J_{Rx} - J_{Px}\right) - \dot{q}_P s_P \left(1 + c_P\right) J_{PTZ} \\ +2\dot{q}_R s_R c_R c_P^2 \left(J_{Ry} - J_{Rz}\right) \end{pmatrix} \\
&-\left(qc_Y - ps_Y\right)\left(pc_Y + qs_Y\right)\begin{pmatrix} J_{Yy} - c_P^2\left(J_{Px} + J_{Rx}\right) + J_{PT} \\ -s_P^2\left(J_{Pz} + J_{Ry} s_R^2 + J_{Rz} c_R^2\right) - J_{Yx} \end{pmatrix} \\
&-\left(r + \dot{q}_Y\right)\left(qc_Y - ps_Y\right)\left(c_P s_P \left(J_{Px} + J_{Rx}\right) - s_P c_P J_{PTZ}\right) \\
&-\left(\left(pc_Y + qs_Y\right)\left(r + \dot{q}_Y\right) c_P c_R s_R - \left(qc_Y - ps_Y\right)^2 s_P c_R^2\right)\times \\
&\left(J_{Ry} - J_{Rz}\right) - c_R s_R s_P \left(pc_Y + qs_Y\right)^2 \left(J_{Ry} - J_{Rz}\right) + J_{Rx}\ddot{q}_R s_R \\
&+J_{Rx}\dot{q}_R \dot{q}_P c_P - \dot{q}_P\left(-\dot{q}_P s_P c_R s_R - \dot{q}_R c_P s_R^2 + \dot{q}_R c_P c_R^2\right)\left(J_{Ry} - J_{Rz}\right)
\end{aligned} \tag{29}
$$

where

$J_{PT} = J_{Py} + J_{Ry} c_R^2 + J_{Rz} s_R^2$, $J_{PTZ} = \left(J_{Pz} + J_{Ry} s_R^2 + J_{Rz} c_R^2\right)$,

and

$J_{YTT} = \left(J_{Yx} + s_P^2\left(J_{Rx} - J_{Px}\right) + c_P\left(pc_Y + qs_Y\right) J_{PTZ}\right)$ is the total moment of inertia of the yaw gimbal about z-axis.

With considering Assumption 2, (29) can be written as

$$
J_{YT}\ddot{q}_Y = t_Y + t_{YI} + t_{YB}, \tag{30}
$$

where

$$
J_{YT} = \left(J_{Yx} + s_P^2\left(J_{Rx} - J_{Px}\right) + c_P\left(pc_Y + qs_Y\right)\left(J_{Pz} + J_{Ry}\right)\right),
$$

$$
\begin{aligned}
t_{YI} &= 2\dot{q}_Y \dot{q}_P c_P s_P \left(J_{Rx} - J_{Px}\right) + \dot{q}_Y \dot{q}_P s_P \left(1 + c_P\right)\left(J_{Py} + J_{Ry}\right) \\
&\quad + J_{Rx}\ddot{q}_R s_R + J_{Rx}\dot{q}_R \dot{q}_P c_P,
\end{aligned} \tag{31}
$$

is the interaction between pitch and roll channels to yaw and

$$
\begin{aligned}
t_{YB} &= -\left(r + \dot{q}_Y\right)\left(qc_Y - ps_Y\right)\left(c_P s_P\left(J_{Px} + J_{Rx}\right) - s_P c_P\left(J_{Pz} + J_{Ry}\right)\right) \\
&-\left(\dot{p}c_Y + \dot{q}s_Y - \dot{q}_Y ps_Y + \dot{q}_Y qc_Y\right)\left(c_P s_P\left(J_{Rx} - J_{Px}\right) + s_P\left(J_{Py} + J_{Ry}\right)\right) \\
&-\left(pc_Y + qs_Y\right)\Big(\left(\dot{q}_P c_P - \dot{q}_P s_P^2\right)\left(J_{Rx} - J_{Px}\right) + \dot{q}_P s_P^2\left(J_{Pz} + J_{Ry}\right) \\
&+\dot{q}_P\left(J_{Py} + J_{Ry}\right)\Big) + \dot{q}_P r s_P\left(1 + c_P\right)\left(J_{Py} + J_{Ry}\right) - \dot{r}J_{YT} \\
&-\left(qc_Y - ps_Y\right)\left(pc_Y + qs_Y\right)\Big(J_{Yy} - c_P^2\left(J_{Px} + J_{Rx}\right) + J_{Py} + J_{Ry} \\
&-s_P^2\left(J_{Pz} + J_{Ry}\right) - J_{Yx}\Big) + 2\dot{q}_P r c_P s_P\left(J_{Rx} - J_{Px}\right),
\end{aligned} \tag{32}
$$

is the effect of base movements to yaw channel.

Remark 3: The effect of base angular velocity ($\boldsymbol{\omega}_{IB}^{B}$) into channels is twofold: direct and indirect effect. As it shown in (18), (25), and (32), some parts don't multiply to $\dot{q}_R$, $\dot{q}_P$, and $\dot{q}_Y$ which are the direct effects. The rest are indirect effects which influence the channels first and then, affect the other channels from the interaction between them.

4. Control Strategy

For stabilizing the sensor in the inertial space, it is necessary to measure the angular velocity of the sensor (frame R) with respect to inertial frame. Therefore, three rate gyroscopes are placed beside the sensor and measure the angular velocity of the sensor in the inertial space ($\boldsymbol{\omega}_{IR}^{R}$). In addition, gimbal rotation angles (q_R, q_P, and q_Y) are measured with resolvers.

Remark 4: Due to mechanical restrictions, the yaw and pitch angles have the following limitations:

$$
\left|q_Y\right| < \frac{p}{2}, \quad \left|q_P\right| < \frac{p}{2} \tag{33}
$$

but the roll angle can rotate freely.

The aim of inertial stabilization of the sensor is to control the angular velocities of gimbal rotations to keep $\boldsymbol{\omega}_{IR}^{R}$ to zero or follow a certain trajectory with respect to platform rotations **Keywords** ($\boldsymbol{\omega}_{IB}^{B}$) and in the presence of disturbances. In this way, for finding $\dot{q}_R$, $\dot{q}_P$, and $\dot{q}_Y$ from $\boldsymbol{\omega}_{IR}^{R}$, the inverse kinematic equations should be solved. Note that there is no need to measure them directly, instead, they can be obtained analytically. According to (7) it is straightforward that

$$
\dot{q}_Y = \frac{q_R s_R + r_R c_R - s_P\left(pc_Y + qs_Y\right)}{c_P} - r, \tag{34}
$$

$$
\dot{q}_P = q_R c_R - r_R s_R - qc_Y + ps_Y, \tag{35}
$$

$$
\dot{q}_R = p_R + s_P\left(r + \dot{q}_Y\right) - c_P\left(pc_Y + qs_Y\right). \tag{36}
$$

Figure 2 shows the control diagram for this system. In this figure, $\boldsymbol{\theta} = \left[q_R \quad q_P \quad q_Y\right]^T$, $\dot{\boldsymbol{\theta}} = \left[\dot{q}_R \quad \dot{q}_P \quad \dot{q}_Y\right]^T$, and $\left(\boldsymbol{\omega}_{IR}^{R}\right)_{ref}$ is the reference signal for tracking problems and usually

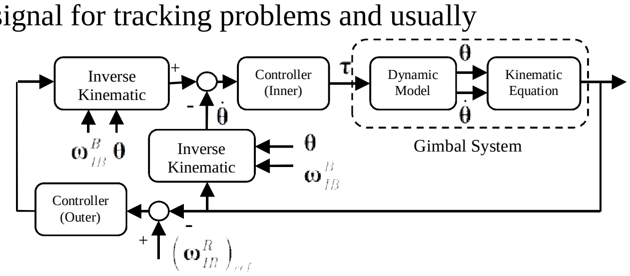

Figure 2: Control diagram

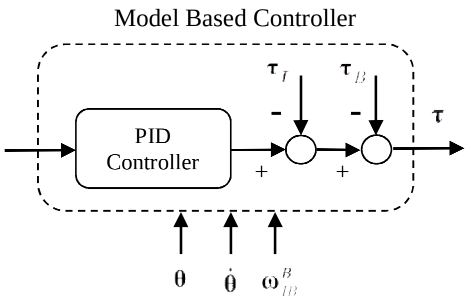

Figure 3: Model based controller for the inner loop control

produced from image processing algorithm or introduced directly by the operator. The inner loop controller is to control the gimbal dynamics, while the outer loop controller is for reducing the tracking error.

The outer loop controller could be a simple PID controller and the inner loop controller could be a model based compensator as it shown in figure 3. In figure 3, $\boldsymbol{\tau}_I = [t_{RI} \quad t_{PI} \quad t_{YI}]^T$ and $\boldsymbol{\tau}_B = [t_{RB} \quad t_{PB} \quad t_{YB}]^T$, where the parameters therein are introduced in (16), (23), and (30).

5. Conclusion

This paper considered the equation of motion derivation for a three axes gimbal system with the help of moment equation. The effect of angular velocities of the base (host vehicle) into the gimbaled dynamic system and cross-coupling between gimbals were presented. Moreover, a control strategy was presented for controlling the gimbal dynamic. Remark 1 and 2 proposed some critical notes for constructing the gimbal assembly.

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