An LMI Approach to Resilient H_∞ Fractional Order Observer Design for Lipschitz Fractional Order Nonlinear Systems Using Continuous Frequency Distribution

Elham Amini Boroujeni [1*], Hamid Reza Momeni[2], Mahdi Sojoodi [3]

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Abstract

Non-fragile H_∞ observer design is the main problem of this paper. Using continuous frequency distribution, the stability conditions based on integer order Lyapunov theorem are derived for Lipschitz class of nonlinear fractional order systems. The proposed observer is stable beside the existence of both gain perturbation and input disturbance. For the first time, in this paper a systematic method is suggested based on linear matrix inequality to find an optimal observer gain to minimize both the effects of disturbance on the synchronization error and norm of the observer gain. A comparison has done between this observer and previous research on resilient H_∞ observer design for nonlinear fractional order systems based on fractional order Lyapunov method. The comparison shows a much broader range of feasible response for the proposed method of this paper besides simpler computing. After presenting thediscussion, chaos synchronization is simulated to show the effectiveness of the proposed method in the end.

Keywords: Continues frequency distribution; H_∞ fractional order observer; Linearmatrix inequality; Lipchitz nonlinear systems; Resilient observer.

1. Introduction

Inspite of ancient history for the fractional order calculus, its applications to physics and engineering have just started in the recent decades [1,2].

Observers are dynamical systems that used to estimate the states of the system from their inputs and outputs, and play an important role in systems control theory and fault detection on dynamical systems. Although there are many researches on the fractional order observer based control of the fractional ordersystem [3-10] but to our best knowledge, almost all of the researches mentioned above and many other existing references mainly have ignored the nonlinearity or linearized it by use of their controller. Sometimes we have no control on the desired system for observation which makes the mentioned researches inadequate. Moreover, the major difficulties in the design of practical observers for dynamical systems are their nonlinear dynamics and existing exogenous disturbances.

According to what was said, observer design for nonlinear fractional order dynamical systems is a widespread area of current researches [11-14]. Fractional algebraic observability property is introduced in [11] as a main tool in the design of an observer structure for a certain autonomous Lipchitz nonlinear fractional order systems. In reference [12] the problem of state estimation for a class of fractional order nonlinear systems with uncertainty, using sliding mode technique is investigated while [13] presented observers design for continuous time singular fractional order systems based on the generalized Sylvester equations solutions.Reference [14] has introduced a nonlinear fractional order observer strategy with unknown input disturbance. This paper has considered a very special model and omitted the disturbance by considering some constraints on the observer coefficients which cause the result don't be operational in many cases.

As introduced in [15], a fragile or non-resilient observer is an observer in which the estimation error diverges by a small perturbation in the observer gain. Since the observer gains are usually obtained from offline calculations, in many practical applications the gain may have slow drifts; thus, it is necessary that the observer tolerates some perturbations in its coefficients [16].

In [17] a specific Lyapunov function is defined and the application of Lyapunov's method to nonlinear Fractional Differential Equations

1 *Assistant Professor, Electrical and Computer Engineering Department, Kharazmi University, Tehran, Iran,: e.amini.b@khu.ac.ir,
2 .Associated Professor, Faculty of Electrical Engineering, Tarbiat Modares University, Tehran, Iran.
3 .Assistant Professor, Faculty of Electrical Engineering, Tarbiat Modares University, Tehran, Iran

(FDEs) is proposed. The key concept of this paper is the frequency distributed fractional integrator model which is the basis of a global state space model of FDEs. Using the mentioned Lyapunov in [17] an integer order Lyapunov based method recommended for stability proof of the nonlinear fractional order systems.Using this method for the first time in reference [18],resilient state estimator introduced for the Lipschitz fractional order systems.The result of [18] is extended to uncertain nonlinear systems in [19] and [20] while [19] has presented an observer based control and a non-fragile observer design for a class of fractional order one sided Lipschitz nonlinear systems is expressed in [20].

Paper [21] is dealt with the design of non-fragile state estimation problem for the Lipschitz fractional order systems based on fractional order Lyapunov technique while [22] has extended the results of [21] and introducedan iterative algorithm to calculate the optimal non-fragile H_∞observer gain for Lipschitz nonlinear fractional order systems based on fractional order Lyapunov theorem.

In this paper we extend the result of [18] to present H_∞non-fragile nonlinear fractional order observer design for Lipschitz nonlinear fractional order systems. The approach is using continues frequency diffusion and presenting stability proof based on the integer order Lyapunov method. Sufficient conditions for robust stability with existence of perturbation in the gain matrix and input disturbance are derived in terms of linear matrix inequalities formulation and unlike [22], a systematic method is introduced to minimize both the effects of disturbance on the synchronization error and norm of the observer gain optimally.

The rest of the paper is organized as follows: Section 2 provides preliminaries. In section 3, the problem statement is given and the optimal gain for the proposed observer is discussed. Comparison between the proposed observer and the results of [22] is investigated and an illustrative simulation of chaos synchronization is provided in section 4 and finally, the conclusion remarks are given in section 5.

2. Preliminaries

In this part the most commonly used definitions are introduced and then diffusive representation that provides the theoretical basis for a time approximation of ${}_aI_t^q f(t)$ is given. Finally we will present some useful Lemmas for our main results.

Definition 1. [23], [24]: The qth-order Riemann-Liouville fractional derivative of function $f(t)$ with respect to t and the initial value a is given by:

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_a^t \frac{f(t)}{(t-t)^{q-m+1}}dt , \quad (1)$$

where m is the first integer larger than$q$and $\Gamma(.)$ is the Gamma function which is defined by $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$.

Remark 1.[22] The q th-order fractional derivative of function $f(x(t))=x(t)^2$ with respect to t is given by

$$ {}_0D_t^q f(x(t)) = x(t){}_0D_t^q x(t) + p_x \quad (2)$$

where

$$ p_x = \sum_{k=1}^{\infty}\frac{\Gamma(1+q)}{\Gamma(1+k)\Gamma(1-k+q)}({}_0D_t^k x)({}_0D_t^{q-k}x) \quad (3)$$

As introduced in [25] and [26], p_x is bounded. So we define the following boundedness condition:

$$ \|p_x\| \le b\,\|x\|^2 \quad (4)$$

To have a smaller b , inequality (4) can modified to (5) while $p_x \in$ and $p_x \le \|p_x\|$:

$$ p_x \le b\,\|x\|^2 \quad (5)$$

Definition 2.[23], [24]:the q th-order fractional integral of function $f(t)$ with respect to t and the initial value a is given by

$$ {}_aD_t^{-q} f(t) = {}_aI_t^q f(t) = \frac{1}{\Gamma(q)}\int_a^t \frac{f(t)}{(t-t)^{1-q}}dt , \quad (6)$$

where $q>0$ and Γ is the Gamma function.

Definition 3. (Diffusive Representation)[17]: Let $h(t)$ be the impulse response of a linear system. The diffusive representation (or frequency weighting function) of $h(t)$ is called $m(w)$ with following relation:

$$ h(t) = \int_0^\infty m(w)e^{-wt}dw \quad (7)$$

Remark 2. [17]: For the fractional order integral ${}_aI_t^q f(t)$, Eq. (6) can be written as:

$$ {}_aI_t^q f(t) = h(t) * f(t) \tag{8} $$

where * denote convolution operator and $h(t)=\dfrac{t^{q-1}}{\Gamma(q)}$. The diffusive representation of $h(t)=\dfrac{t^{q-1}}{\Gamma(q)}$ is introduced as:

$$ m(w)=\frac{\sin(qp)}{p}w^{-q} \tag{9} $$

Definition 4.[17]: The nonlinear FDE:

$$ {}_aD_t^q x = f(x) \tag{10} $$

due to the continuous frequency distributed model of the fractional integrator, can be expressed as:

$$ \begin{cases} \dfrac{\partial z(w,t)}{\partial t} = -w\,z(w,t)+f(x(t)) \\ x(t)=\displaystyle\int_0^\infty m(w)z(w,t)dw \end{cases} \tag{11} $$

WHILE $m(w)$ IS THE SAME AS (9).

Lemma 1. (Schur Complement) [27]: for a real matrix $\Sigma=\Sigma^T$, the following assertions are equivalent:

$$ \begin{aligned} &1)\ \Sigma=\begin{bmatrix} Q(x) & W(x) \\ W^T(x) & R(x)\end{bmatrix}<0 \\ &2)\ R(x)<0,\quad Q(x)-W(x)R^{-1}(x)W^T(x)<0 \\ &3)\ Q(x)<0,\quad R(x)-W^T(x)Q^{-1}(x)W(x)<0 \end{aligned} \tag{12} $$

Lemma 2. [28]: Let *x*, *y* be real vectors of the same dimension. Then, for any scalar $e>0$, we have:

Lemma 3. [29]: The design of the robust proportional observer consists in finding a matrix *L* such as the estimation error, $\tilde{x}(t)$, satisfies the following H_∞ performances:

WHERE h IS THE L_2 GAIN FROM DISTURBANCE, $w(t)$, TO $\tilde{x}(t)$ TO BE MINIMIZED.

3. Robust Resilient Nonlinear Fractional Order Observer Design

Consider a nonlinear fractional order system of the form:

$$ \begin{aligned} &{}_aD_t^q x = Ax + j(x,u)+w \\ &y = Cx \end{aligned} \tag{15} $$

where $0<q<1$ and $x\in{}^n$, $u\in{}^k$, $y\in{}^m$ are the state, input, and output respectively. $A\in{}^{n\times n}$ and $C\in{}^{m\times n}$ are constant matrixes, $w\in{}^n$ is the input disturbance and $j:[{}^n\ \ {}^k]\rightarrow{}^n$ is nonlinear function which is Lipschitz in x with Lipschitz constant $g>0$, i.e. :

$$ \|j(x_1,u)-j(x_2,u)\|<g\,\|x_1-x_2\| \tag{16} $$

Let the resilient nonlinear fractional order observer be expressed as:

$$ \begin{aligned} &{}_0D_t^q\hat{x}=A\hat{x}+j(\hat{x},u)+(L+\Delta(t))(y-C\hat{x}) \\ &\hat{y}=C\hat{x} \end{aligned} \tag{17} $$

where $\hat{x}\in{}^n$ is the state estimation and $L\in{}^{n\times m}$ isthe proportional observer gain and the terms $\Delta(t)\in{}^{n\times m}$ is additive perturbation on the error gain with known bound $\|\Delta(t)\|\le r$. The observer error dynamic equation is obtained as:

$$ \begin{aligned} {}_0D_t^q\tilde{x} &= (A-LC-\Delta(t)C)\tilde{x} \\ &\quad + j(x,u)-j(\hat{x},u)+w \end{aligned} \tag{18} $$

where $\tilde{x}=x-\hat{x}$ is the state estimation error.

The following theorem provides sufficient conditions for the stability of the resilient fractional order observer (17).

Theorem: Consider the resilient observer (17). This observer has a stable observation if

$$ 2x^Ty\le e\,x^Tx+e^{-1}y^Ty \tag{13} $$

positive real numbers e_1,e_2 and matrice $P>0$ exist while the proportional observer gain is the solution of the following constrained LMI:

$$ \begin{aligned} &\lim_{t\to\infty}\tilde{x}(t)=0 \qquad for\ w(t)=0 \\ &\|\tilde{x}(t)\|_2^2\le h\|w(t)\|_2^2 \qquad for\ w(t)\neq 0\ and\ \tilde{x}(0)=0 \end{aligned} \tag{14} $$

$$ \begin{bmatrix} \Gamma & \dfrac{P}{2} & P^T & P^T \\ \dfrac{P^T}{2} & -hI & 0 & 0 \\ P & 0 & -2e_1I & 0 \\ P & 0 & 0 & -2e_2I \end{bmatrix}<0 \tag{19} $$

in which

$$\Gamma = \frac{PA + A^T P^T}{2} - \frac{SC + C^T S^T}{2} + \frac{e_2 r^2}{2} C^T C + \frac{e_1}{2} g^2 I + I$$

h **is** the L_2 gain from disturbance to error as introduced in (14), $g > 0$ is Lipschitz constant of nonlinear function in system (15) while r is the known bound of the additive perturbation on the error gain, $S = PL$. So the proportional observer gain is equal to $L = P^{-1} S$.

Proof: Using definition 4, (18) can be written as:

$$\begin{cases} \dfrac{\partial z(w,t)}{\partial t} = -w z(w,t) + (A - LC - \Delta(t)C)\tilde{x} \\ \qquad + j(x,u) - j(\hat{x},u) + w \\ \tilde{x}(t) = \displaystyle\int_0^\infty m(w) z(w,t) dw \end{cases} \tag{20}$$

where $m(w)$ is the same as (9).

Let us define two Lyapunov functions: $n(w,t)$ is the monochromatic Lyapunov function corresponding to the elementary frequency w and $V(t)$ is the Lyapunov function summing all the monochromatic $n(w,t)$ with the weighting function $m(w)$. Now we define our monochromatic Lyapunov function such as:

$$n(w,t) = \frac{z^T(w,t) P\, z(w,t)}{2}, \qquad P \in {}^{n\times r} \tag{21}$$

this definition results in:

$$\frac{\partial n(w,t)}{\partial z(w,t)} = z^T(w,t) P \tag{22}$$

and

$$\begin{aligned} \frac{\partial n(w,t)}{\partial t} &= \frac{\partial n(w,t)}{\partial z(w,t)} . \frac{\partial z(w,t)}{\partial t} \\ &= z^T(w,t) P . (-w z(w,t) + j(x,u) - j(\hat{x},u) \\ &\quad + (A - LC - \Delta(t)C)\tilde{x} + w) \end{aligned} \tag{23}$$

According to the definition of $V(t)$ we have:

$$V(t) = \int_0^\infty m(w) n(w,t) dw \tag{24}$$

Taking the derivative of Eq. (24) causes:

$$\frac{dV(t)}{dt} = \int_0^\infty m(w) \frac{\partial n(w,t)}{\partial t} dw \tag{25}$$

Using Eq.(21) and (24) concludes that:

$$V(t) = \frac{1}{2} \int_0^\infty m(w) z^T(w,t) P\, z(w,t) dw \tag{26}$$

and substituting Eq. (23) into (25) follows that:

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_0^\infty m(w) \Big(z^T(w,t) P . (-w z(w,t) + (A - LC - \Delta(t)C)\tilde{x} + j(x,u) - j(\hat{x},u)) + w \Big) dw \\ &= \int_0^\infty m(w) z^T(w,t) dw . P((A - LC - \Delta(t)C)\tilde{x} + j(x,u) - j(\hat{x},u) + w) \\ &\quad - \int_0^\infty w\, m(w) z^T(w,t) P z(w,t) dw \end{aligned} \tag{27}$$

Using Eq. (20) simplifies (27) as follow:

$$\begin{aligned} \frac{dV(t)}{dt} &= \tilde{x}^T P[(A - LC - \Delta(t)C)\tilde{x} \\ &\quad + j(x,u) - j(\hat{x},u) + w] \\ &\quad - \int_0^\infty w\, m(w) z^T(w,t) P z(w,t) dw \end{aligned} \tag{28}$$

According to the Lyapunov theorem, stability conditions of the considered system are $V(t) > 0$ and $\frac{dV(t)}{dt} < 0$. Using (26) and (28) implies that $V(t)$ is positive if $P > 0$ and $\frac{dV(t)}{dt}$ is negative if

$$\begin{aligned} &\tilde{x}^T P[(A - LC - \Delta(t)C)\tilde{x} \\ &\qquad + j(x,u) - j(\hat{x},u) + w] < 0 \end{aligned} \tag{29}$$

or equivalently

$$\tilde{x}^T P(A-LC-\Delta(t)C)\tilde{x} + \tilde{x}^T P[j(x,u)-j(\hat{x},u)] + \tilde{x}^T Pw < 0 \tag{30}$$

Using Lemma 2 on the second term, with e_1 result in:

$$\begin{aligned}&\tilde{x}^T P(A-LC-\Delta(t)C)\tilde{x} + \frac{1}{2e_1}\tilde{x}^T P^T P\tilde{x} + \tilde{x}^T Pw \\ &+\frac{e_1}{2}(j(x,u)-j(\hat{x},u))^T(j(x,u)-j(\hat{x},u)) \\ &\le \tilde{x}^T P(A-LC-\Delta(t)C)\tilde{x} + \frac{1}{2e_1}\tilde{x}^T P^T P\tilde{x} \\ &+\frac{e_1}{2}(\|j(x,u)-j(\hat{x},u)\|^2 + \tilde{x}^T Pw < 0\end{aligned} \tag{31}$$

Applying inequality (16) to inequality (31) concludes that:

$$\begin{aligned}&\tilde{x}^T P(A-LC-\Delta(t)C)\tilde{x} + \frac{1}{2e_1}\tilde{x}^T P^T P\tilde{x} \\ &+\frac{e_1}{2}g^2\|\tilde{x}\|^2 + \tilde{x}^T Pw \\ &\le \tilde{x}^T P(A-LC-\Delta(t)C)\tilde{x} + \frac{1}{2e_1}\tilde{x}^T P^T P\tilde{x} \\ &+\frac{e_1}{2}g^2\tilde{x}^T\tilde{x} + \tilde{x}^T Pw < 0\end{aligned} \tag{32}$$

The inequality (32) can be rewritten as bellow.

$$\begin{aligned}&\tilde{x}^T P(A-LC)\tilde{x} - \tilde{x}^T P\Delta(t)C\tilde{x} \\ &+\frac{1}{2e_1}\tilde{x}^T P^T P\tilde{x} + \frac{e_1}{2}g^2\tilde{x}^T\tilde{x} + \tilde{x}^T Pw < 0\end{aligned} \tag{33}$$

Again using Lemma 2 on the second term, with e_2 result in:

$$\begin{aligned}&\tilde{x}^T P(A-LC)\tilde{x} + \frac{1}{2e_2}x^T P^T P\tilde{x} \\ &+\frac{e_2}{2}\tilde{x}^T C^T[\Delta(t)]^T\Delta(t)C\tilde{x} + \frac{1}{2e_1}\tilde{x}^T P^T P\tilde{x} \\ &+\frac{e_1}{2}g^2\tilde{x}^T\tilde{x} + \tilde{x}^T Pw < 0\end{aligned} \tag{34}$$

While $\|\Delta(t)\| \le r$ inequality (34) can be written as follow:

$$\begin{aligned}&\tilde{x}^T P(A-LC)\tilde{x} + \frac{1}{2e_2}x^T P^T P\tilde{x} \\ &+\frac{e_2 r^2}{2}\tilde{x}^T C^T C\tilde{x} + \frac{1}{2e_1}\tilde{x}^T P^T P\tilde{x} \\ &+\frac{e_1}{2}g^2\tilde{x}^T\tilde{x} + \tilde{x}^T Pw < 0\end{aligned} \quad ((35$$

Using Lemma 3, when $w=0$ the sufficient conditions for observer convergence can be obtained by setting:

$$\begin{aligned}&\tilde{x}^T[P(A-LC) + (\frac{1}{2e_2}+\frac{1}{2e_1})P^T P \\ &+\frac{e_1}{2}g^2 I + \frac{e_2 r^2}{2}C^T C]\tilde{x} < 0\end{aligned} \tag{36}$$

Inequality (36) is not necessarily a symmetric matrix. Thus it cannot be converted to LMI by using Lemma 1. To overcome this problem we introduce $Q = PA - SC$ in which $S = PL$ and rewrite (36)as:

$$\begin{aligned}&\tilde{x}^T[(\frac{1}{2e_2}+\frac{1}{2e_1})P^T P + \frac{e_1}{2}g^2 I + \frac{e_2 r^2}{2}C^T C]\tilde{x} \\ &+\tilde{x}^T[\frac{Q+Q^T}{2} + \frac{Q-Q^T}{2}]\tilde{x} < 0\end{aligned} \quad (3$$

Since $\tilde{x}^T Q\tilde{x} \in$ and $(\tilde{x}^T Q\tilde{x})^T = \tilde{x}^T Q^T\tilde{x}$, we have:

$$(\tilde{x}^T Q\tilde{x})^T = \tilde{x}^T Q\tilde{x} = \tilde{x}^T Q^T\tilde{x} \tag{38}$$

This implies that:

$$\tilde{x}^T(\frac{Q-Q^T}{2})\tilde{x} = 0 \tag{39}$$

Substituting (39) into (37) follows that:

$$\begin{aligned}&\tilde{x}^T(\frac{Q+Q^T}{2})\tilde{x} + \tilde{x}^T[(\frac{1}{2e_2}+\frac{1}{2e_1})P^T P \\ &+\frac{e_1}{2}g^2 I + \frac{e_2 r^2}{2}C^T C]\tilde{x} \\ &\le \tilde{x}^T[\frac{Q+Q^T}{2} + (\frac{1}{2e_2}+\frac{1}{2e_1})P^T P \\ &+\frac{e_1}{2}g^2 I + \frac{e_2 r^2}{2}C^T C]\tilde{x} < 0\end{aligned} \tag{40}$$

Therefore the new conditions of our Lyapunov candidate are as bellow:

$$\begin{aligned}&P>0, \\ &\frac{Q+Q^T}{2} + (\frac{1}{2e_2}+\frac{1}{2e_1})P^T P + \frac{e_1}{2}g^2 I + \frac{e_2 r^2}{2}C^T C < 0\end{aligned} \tag{41}$$

Using lemma 3 for H_∞ observer we have:

$$\tilde{x}^T(t)\tilde{x}(t)-hw^T(t)w(t)\le 0 \tag{42}$$

On the other hand, for stability of the observer it is necessary to have $\frac{dV(t)}{dt}<0$ which causes:

$$\tilde{x}^T\tilde{x}-hw^Tw+\frac{dV(t)}{dt}<0 \tag{43}$$

Using
(28) and ((35)we have:

$$\begin{aligned}\frac{dV(t)}{dt}=&\tilde{x}^TP(A-LC)\tilde{x}+\frac{1}{2e_2}x^TP^TP\tilde{x}\\&+\frac{e_2r^2}{2}\tilde{x}^TC^TC\tilde{x}+\frac{1}{2e_1}\tilde{x}^TP^TP\tilde{x}+\frac{e_1}{2}g^2\tilde{x}^T\tilde{x}\\&+\tilde{x}^TPw-\int_0^\infty w\,m(w)z^T(w,t)Pz(w,t)dw\end{aligned} \tag{44}$$

Inequality (43) can be rewritten as:

$$\begin{aligned}&\tilde{x}^T\tilde{x}-hw^Tw+\tilde{x}^TP(A-LC)\tilde{x}\\&+\frac{1}{2e_2}x^TP^TP\tilde{x}+\frac{e_2r^2}{2}\tilde{x}^TC^TC\tilde{x}\\&+\frac{1}{2e_1}\tilde{x}^TP^TP\tilde{x}+\frac{e_1}{2}g^2\tilde{x}^T\tilde{x}+\tilde{x}^TPw\\&-\int_0^\infty w\,m(w)z^T(w,t)Pz(w,t)dw<0\end{aligned} \tag{45}$$

Using **Error! Reference source not found.** for symmetrization, (45) can be replaced by:

$$\begin{aligned}&\tilde{x}^T[\frac{Q+Q^T}{2}+(\frac{1}{2e_2}+\frac{1}{2e_1})P^TP+\frac{e_1}{2}g^2I\\&+\frac{e_2r^2}{2}C^TC+I]\tilde{x}-hw^Tw+\tilde{x}^TPw\\&-\int_0^\infty w\,m(w)z^T(w,t)Pz(w,t)dw<0\end{aligned} \tag{46}$$

Summarizing (46), the sufficient condition can be offered as:

$$\begin{aligned}&\tilde{x}^T[\frac{Q+Q^T}{2}+(\frac{1}{2e_2}+\frac{1}{2e_1})P^TP+\frac{e_1}{2}g^2I\\&+\frac{e_2r^2}{2}C^TC+I]\tilde{x}-hw^Tw+\tilde{x}^TPw<0\end{aligned} \tag{47}$$

or

$$X^T\begin{bmatrix}\Pi & \frac{P}{2}\\ \frac{P^T}{2} & -hI\end{bmatrix}X<0 \tag{48}$$

Where

$$\Pi=\frac{Q+Q^T}{2}+(\frac{1}{2e_2}+\frac{1}{2e_1})P^TP+\frac{e_1}{2}g^2I+\frac{e_2r^2}{2}C^TC+I$$

$X=[\tilde{x}\quad w]^T$ and $\tilde{x}^TPw=\left(\tilde{x}^TPw\right)^T\in$ .This can be altered to LMIs by using Schur compliment as follow:

$$\begin{pmatrix}\Gamma & \frac{P}{2} & P^T & P^T\\ \frac{P^T}{2} & -hI & 0 & 0\\ P & 0 & -2e_1I & 0\\ P & 0 & 0 & -2e_2I\end{pmatrix}<0 \tag{49}$$

while $\Gamma=\frac{Q+Q^T}{2}+\frac{e_2r^2}{2}C^TC+\frac{e_1}{2}g^2I+I$ and $P>0$. Note that (41)and (47)are included in(49). □

Remark 3. The proposed LMI is consist of h. Thus, depend on this parameter, which can be a fixed constant or an optimization or minimization variable, the observer design is an optimization or minimization problem or an LMI feasibility problem.

Remark 4. To solve LMI (19) optimally, a tradeoff between amount of h and $\|L\|$ should be considered.While $L=P^{-1}S$, we have $\|L\|\le\|P^{-1}\|\cdot\|S\|$ andwe should minimize $\|P^{-1}\|$ and $\|S\|$ to decrease $\|L\|$. The inequality (50) is introduced to determine the upper bound of $\|L\|$.

$$\left.\begin{aligned}\|S\|<k_s\Rightarrow\begin{bmatrix}K_sI & S^T\\ S & K_sI\end{bmatrix}>0\\ \|P^{-1}\|<k_p\Rightarrow\begin{bmatrix}P & I\\ I & K_pI\end{bmatrix}>0\end{aligned}\right\}\Rightarrow\|L\|\le k_p\cdot k_s \tag{50}$$

Using $\min(w_1\cdot h+w_2\cdot k_p+w_3\cdot k_s)$ subject to (19) will minimize both h and $\|L\|$ besides solving inequality (19) by considering weight coefficients.

4. SIMULATION RESULTS AND DISCUSSION

Robust and non-fragile observer design for nonlinear fractional order systems is recently presented in [22] based on fractional order Lyapunov theorem. In continue; we present the result of this paper and compare the result of designing robust non-fragile observer based on fractional and integer order Lyapunov theorem.

According to [22], the non-fragile nonlinear observer (17) has a robust observation; regardless the disturbance affects the system (15), if positive real numbers e_{21}, e_{22} and $P_2 \in {}^{n\times n}$ exists while the proportional observer gain is the solution of the following optimization problem:

Optimize h_2 and L_2

subject to $P_2 > 0$ and

$$\begin{bmatrix} \Gamma_2 & \frac{P_2}{2} & P_2^T & P_2^T \\ \frac{P_2^T}{2} & -h_2 I & 0 & 0 \\ P_2 & 0 & -2e_{21}I & 0 \\ P_2 & 0 & 0 & -2e_{22}I \end{bmatrix} < 0 \tag{51}$$

While $S_2 = P_2 L_2$ and the propo

$$\Gamma_2 = \frac{P_2A + A^T P_2^T}{2} - \frac{S_2C + C^T S_2^T}{2} + \frac{e_{22}r^2}{2}C^T C + \frac{e_{21}}{2}g^2 I + bP_2 + I$$

rtional observer gain $L_2 = P_2^{-1}S_2$ stabilizes the state estimation error (18) for all gain perturbation satisfying $\|\Delta(t)\| \le r$. Parameter b is defined in (4)and as it has explained in [22], inequality (51)should be solved in an iterative algorithm to find a proper b besides solving (51).

Comparing inequality (51)which is the result of the designing fractional order Lyapunov based non-fragile robust observer and inequality (19) which indicate the result of this paper in stability proof of robust nonfragile H_∞ with the help of integer order Lyapunov; it shows the similar results for both methods. The only difference is in column 1, row 1 of the main inequality which appears with existence of bP_2 and caused due to simplification of continuous frequency distributed model of the fractional integrator. Although existence of $bP_2 > 0$ shows the more accuracy of fractional order Lyapunov based method, but feasibility region becomes less. In continue, we design and simulate non-fragile H_∞ observer(17) for a chaotic system to illustrate the performance of the integer order Lyapunov based design of robust non-fragile observer and compare it with fractional order Lyapunov based one.

Consider a Lipschitz nonlinear fractional order system with four scroll attractors as introduced in [xxx] where $x = [x_1, x_2, x_3]^T$, $q = 0.85$, $(a,b,c,d,h) = (4, 2.7, 3, 5, 4)$ and g is chosen equal to 0.6 .

$$D^q x = \begin{bmatrix} -a & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & -h \end{bmatrix} x + \begin{bmatrix} bx_2x_3 \\ -x_1x_3 \\ dx_1x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 2 \\ 3 \end{bmatrix} \sin(4t) \tag{52}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Using YALMIP toolbox [xxxi], Ninteger toolbox [xxxii] and LMI control toolbox [xxxiii] in MATLAB and solving LMI (19) besides considering $e_1, e_2 > 0, P = P^T > 0$ and $r = 0.3$, solution for different $W_i, i = 1,2,3$ is derived as table I:

Table I. solution of LMI (19) for different $W_i, i = 1,2,3$.

W_1	W_2	W_3	h	K_p	K_s	$\|L\|$
1	0	0	6.28	6.9254 e+006	3.0802 e+007	2.9647 e+008
1	1	1	64.59	18.80	70.072	1.2340 e+003
10	1	1	34.03	38.01	171.72	2.5297 e+003
100	1	1	16.29	82.41	687.86	6.0536 e+003
300	1	1	12.72	122.6	1269.1	1.0449 e+004

The first row of table I show the results of usual minimization solution of LMI (19) for h . As it has shown, h has the minimum amount equal to 6.28 to make LMI (19) feasible for system (52). Because of the big amount of $\|L\|$ in the minimization method, it is better to use optimal method which has shown in subsequently rows.

According to remark 4, equation $w_1 \cdot h + w_2 \cdot k_p + w_3 \cdot k_s$ should be minimized. We use trial and error to choose the best w_i due to the importance of h besids logical amount of $\|L\|$.

Choosing $w_1 = 300, w_2 = 1, w_3 = 1$ causes the solution to be derived as:

$S = [1268.4, \quad 40.8, \quad -14.5]^T, e_1 = 10.4283, e_2 = 1470,$

$, P = \begin{bmatrix} 165.4137 & -5.7969 & -0.3911 \\ -5.7969 & 0.2113 & -0.0185 \\ -0.3911 & -0.0185 & 58.1251 \end{bmatrix}$. Hence, the observer gain is obtained as $L = [374, \quad 10442, \quad 6]^T$.

To simulate the proposed observer, gain perturbation is considered equal to $\Delta(t) = [0.24, \ 1.5, \ 0.3]^T.\sin(4t)$ while input disturbance is equivalent to $w(t) = [0.25\sin(3t), \quad 0.1u(t), \quad \frac{1}{t+1}]^T$ and initial values for master chaotic system are considered as $[0, 2, 8]$.

Real and estimated values of statesare plotted in fig 1 and estimation errors for the proposed non-fragile H_∞ fractional order nonlinear observer is shown in fig 2.

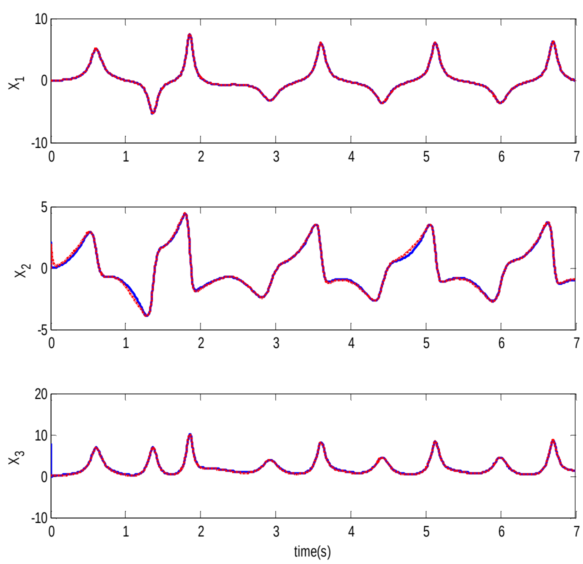

Figure 1. Real value of states (solid line) and estimated value of states (dashed line).

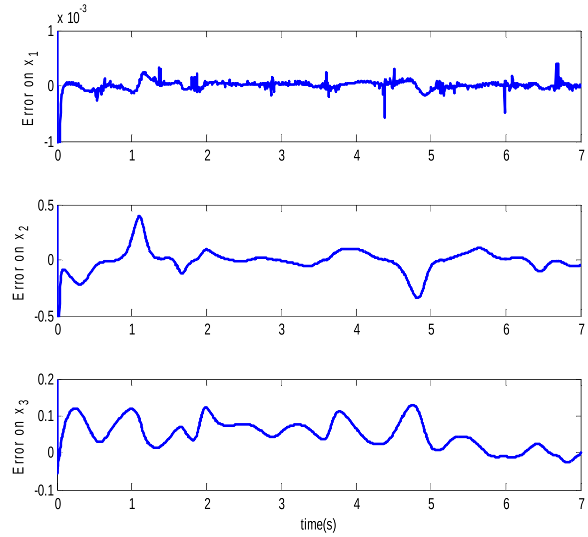

Figure 2. Resilient H_∞ nonlinear observer estimation's error.

It can be seen that using the proposed observer, the slave system can effectively track master chaotic system.Due to existance of disturbance and gain perturbation and $h = 12.7262$, the estimation error is proper.

To have a comparison with the result of [22], we tried to solve (51) with an iterative algorithm for system (52). Figure 3 shows the feasible region of LMI (51) for system (52) with the norm of observer gain in terms of b . As it can be seen b should be less than 3.1 to make (51) feasible which is not accessible for system (52) with iterative algorithm mentioned in [22].

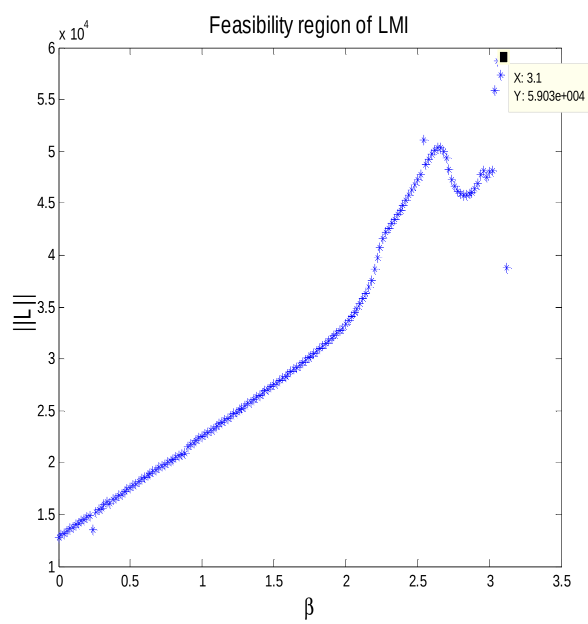

Figure 3. The feasible region of LMI (51) for system (52).

As it was explained before, fractional order Lyapunov based design is more accurate than integer order Lyapunov based design but it has more restrictions which causes less feasible region for this method.

5. Conclusion

This paper presentsa systematic algorithm for designing an optimal H_∞ resilient fractional order observer for a class of nonlinear fractional order systems.Using continuous frequency distribution, the stability conditions based on integer order Lyapunov theorem are derived and converted to LMIs to systematically minimize both the effects of disturbance on the synchronization error and norm of the observer gain.This observer ensures the state estimates converge to its true value in the presence of exogenous disturbances input and observer gain perturbation.

A comparison has done between this observer and fractional order Lyapunov based resilient H_∞ observer design for nonlinear fractional order systems. This investigation showsbigger feasibility region for the integer order Lyapunov based design besides simpler computing.The effectiveness of the proposed observer has indicated through chaos synchronization.

1. Reference

[1] PL. Butzer, U. Westphal, An introduction to fractional calculus, World Scientific, Singapore, 2000.

[2] R. Hilfer, Application of fractional calculus in physics, World Scientific, New Jersey, 2001.

[3] D. Matignon, B. D. Novel, Obsever-Based Controllers for fractional differential systems, Proc of the 36th conference on Decision & Control, San Diego, California USA, 1997, 4967-4972.

[4] Y.H. Lan, H.-X. Huang, Y. Zhou, Observer-based robust control of a ($1< a <2$) fractional-order uncertain systems: a linear matrix inequality approach, IET Control Theory Appl. 6, 2012, 229–234.

[5] M. Feki, R. Gain, Synchronizing fractional order chaotic systems with an integer order observer , 8th International Multi-Conference on Systems, Signals & Devices, Sousse, 2011, 1-5.

[6] S.H. Hosseinnia, R. Ghaderi, A. Ranjbar , M. Mahmoudiana, S. Momani, Sliding mode synchronization of an uncertain fractional order chaotic system, Computers and Mathematics with Applications, 59, 2010, 1637-1643.

[7] L. Xinjie, L. Jie, D. Pengzhen, X. Lifen, Observer Designing for Generalized Synchronization of Fractional Order Hyper-chaotic Lu System, Chinese Control and Decision Conference, China, 2009, 426-431.

[8] P. Zhou, and X. Cheng, Synchronization between different fractional order chaotic systems, Proc. 7th World Congress on Intelligent Control and Automation, Chongqing, China, 2008, 4659-4662.

[9] A. Kiani, K. Fallahi, N. Pariz, H. Leung, A chaotic secure communication scheme using fractional chaotic systems based on an extended fractional Kalman filter, Commun Nonlinear Sci Numer Simulat., 14, 2009, 863-879.

[10] P. Zhou, W. Zhu., Function projective synchronization for fractional-order chaotic systems, Nonlinear Analysis: Real World Applications,12, 2011, 811-816.

[11] R. M. Martinez, J. L. Mata-Machuca, R. M. Guerra, J. A. Leon, G. Fernandez-Anaya, A New Observer for Nonlinear Fractional Order Systems, In Proc: 50th IEEE Conference on Decision and Control and European Control Conf., Orlando, FL, USA, 2011, 3319 - 3324.

[12] S. Dadras, H. R. Momeni, Fractional Sliding Mode Observer Design for a Class of Uncertain Fractional Order Nonlinear Systems, In Proc: 50th IEEE Conference on Decision and Control and European Control Conf., Orlando, FL, USA , 2011, 6925 - 6930.

[13] I. N'Doye, M. Darouach, M. Zasadzinski, N. E. Radhy, Observers Design for Singular Fractional-Order Systems, In Proc: 50th IEEE Conference on Decision and Control and European Control Conf., Orlando, FL, USA, 2011, 4017-4022.

[14] I. N'Doye, H. Voos, M. Darouach, J.G. Schneider, N. Knauf, An Unknown Input Fractional-Order Observer Design for Fractional Order Glucose-Insulin System, In Proc: IEEE EMBS International Conference on Biomedical Engineering and Sciences, Hong Kong, China, 2012,595-600 .

[15] C. S. Jeong, E. E. Yaz, A. Bahakeem, Y. I. Yaz, Resilient design of observers with general criteria using LMIs, In Proc: American Control Conference, Minnesota, USA , 2006.

[16] M. Pourgholi, V. J. Majd, A novel robust proportional-integral (PI) adaptive observer design for chaos synchronization, Chin. Phys. B. , 2011, 20120503.

[17] J.C. Trigeassou, N.Maamri, J.Sabatier, A.Oustaloup, A Lyapunov approach to the stability of fractional differential equations, Signal Process., 91, 2011, 437–445.

[18] E. A. Boroujeni,H. R. Momeni, Non-fragile nonlinear fractional order observer design for a class of nonlinear fractional order systems, Signal Process. 92, 2012, 2365–2370.

[19] C. Li, J. Wang, J.Lu, Observer-based robust stabilisation of a class of non-linear fractional-order uncertain systems: an linear matrix inequalitie approach, IET Control Theory Appl., 6, 2012, 2757–2764.
[20] Y. H. Lan,W. J. Li, Y. Zhou,Y.P. Luo, Non-fragile Observer Design for Fractional-order One-sided Lipschitz Nonlinear Systems , Int. J Automation and Computing, 10, 2013, 296-302.
[21]E. A. Boroujeni,H. R. Momeni,Lyapunov Based Design of a Nonlinear Resilient Fractional Order Observer, In Proc: 21st Iranian Conference on Electrical Engineering, Mashhad, Iran, 2013.
[22] E. Amini Boroujeni, H. R Momeni, An Iterative Method to Design Optimal Non-Fragile H_∞ Observer for Lipschitz Nonlinear Fractional Order Systems, Nonlinear Dynamics, 80, 2015, 1801-1810.
[23] S. Das, Functional Fractional Calculus for System Identification and Controls, 1st Edition, Springer Berlin Heidelberg New York, 2008.
[24] I. Podlubny, Fractional differential equations. Academic Press, New York, 1999.
[25] M. O. Efe: Fractional Fuzzy Adaptive Sliding-Mode Control of a 2-DOF Direct-Drive Robot Arm, IEEE T Syst. Man Cy. B, 38,2008, 1561-1570.
[26] M. O. Efe,C. Kasnakoglu: A fractional adaptation law for sliding mode control, Int. J. Adapt. Control, 22, 2008, 968-986.
[27] S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1994.
[28] F. Chen, W. Zhang, LMI criteria for robust chaos synchronization of a class of chaotic systems, Nonlinear Analysis TMA., 67, 2007, 3384-3393.
[29] R. Orjuela, B. Marx, J. Ragot and D. Maquin, Design of robust H_∞ observers for nonlinear systems using a multiple model, Proc of the 17th World Congress, The International Federation of Automatic Control , 2008, 2756-2761.
[xxx] L. G.Yuan, Q. G. Yang, Parameter identification and synchronization of fractional-order chaotic systems, Commun Nonlinear Sci Numer Simulat,17, 2012, 305–16.
[xxxi] J. Lofberg, YALMIP: A toolbox for modeling and optimization in MATLAB, IEEE Int. Symp. Computer Aided Contol Syst., 2004, 284-89.
[xxxii] D. Valério, Ninteger V. 2.3 Fractional control toolbox for Matlab, Fractional derivatives and applications, User and programmer manual, Technical University of Lisboa. 2005.
[xxxiii] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilai, LMI control toolbox user’s guide, Massachusetts: The Mathworks,1995.