Spherical Trigonometry, a Bridge from Trajectory Waypoints to Guidance Algorithm

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Abstract

Conventionally flight path in airplanes and unmanned air vehicles is determined with waypoints. Waypoints are points on the surface of the earth with specific latitude and longitude. For accurate crossing the waypoints at a specific time, definition of accurate guidance error parameters is essential. Guidance algorithm based on these parameters can make appropriate commands. In this article two parameters, guidance latitude and guidance longitude, based on spherical trigonometry, are defined. Indeed these parameters show guidance error in horizontal channel and longitudinal channel respect to great circles between waypoints. These parameters can be calculated in a closed form and solution of complicated integrals, which is in geodetics on an ellipsoid, do not required. Also guidance algorithms in two channel based on these parameters are designed. In horizontal channel, a PD controller and in longitudinal channel a proportional controller on the difference between desired and real velocity, are designed as guidance algorithms. Also performance of these algorithms is shown with simulation results in comparison with plane simulation.

Keywords: Spherical trigonometry, Guidance latitude, Guidance longitude, Waypoints, Guidance algorithm,

1. Introduction

In planes and UAVs' motion, flight path from initial to final point is defined by waypoints. In some articles, these waypoints are three dimensional and, with arrival time are defined as four dimensional [1,2]. But in conventional structure, with separating horizontal and vertical channel, waypoints are expressed with latitude and longitude to determine trajectory in horizontal plane and as two dimensional [3].

These waypoints usually are determined based on the position of navaids. Navaid is a transmitter on the earth that send waves around itself. Plane gives these waves by a receiver and according to the type of navaid, recognizes itself distance or motion direction respect to navaid [4]. But in some cases; like unpredictable conditions or special flight missions; waypoints are located on a place that is not the place of a navaid.

To guide UAV through these waypoints, it is needed to clearly define UAV orientation respect to these waypoints and their connecting arc. In the presence of radio navaids, these guidance parameters are calculated directly using receivers. But in UAVs or planes, it is desire not to limit trajectory waypoints to navaids. In this case, calculating UAV distance from desired trajectory or calculating remained distance to active waypoint is important in making proper guidance commands in lateral and longitudinal channel.

During years, several works has been done in calculating distances and angles on ellipsoid earth. In this field, some works have been done by Bessel [5], Legendre [6] and Oriani [7] and until recent years different works has been done [8,9]. But obtaining an accurate values for these parameters is not possible in a closed form solution and needs solving a lot of integral relations, so it will be hard to use them in a practical case. In the first part of the article, one way of calculating distance and arc angle on an ellipsoid; which is in [8]; is mentioned. In the other hand, by calculation of arc length and angles between them on a sphere, they could reach a closed form solution in the format of spherical trigonometry. Large part of the way of obtaining these answers is accomplished by Muslim scholars like Khwarizmi in ninth century and Buzjani in tenth century.

The present article uses spherical trigonometry point of view and shows that by this structure, calculation of trajectory parameters and guidance errors don't need any solving of complicated integrals and can be obtained in the format of a simple and accurate relations. These guidance parameters are calculated during movement from trajectory waypoints and, they are latitudinal and longitudinal errors in trajectory passing. Also, to show the applicability of obtainable parameters

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from this structure, a guidance algorithm in lateral and longitudinal channel is proposed that send a proper guidance command to UAV using these parameters. Also, in this case, inaccuracy value in desired trajectory passing, if we use the corresponding parameters in plane geometry, is shown.

In following, a way for calculating length and angle of the trajectory between two waypoints on an ellipsoid is proposed. Then, an introduction about spherical trigonometry as a way that is used in this paper is proposed according to [10] and after it, guidance latitude and longitude parameters are defined as guidance index in lateral and longitudinal channel. Because of the high similarity between these two parameters and geocentric latitude and longitude, they are entitled as guidance latitude and longitude. In the next step, the way of using these parameters to guide UAV is expressed. In this phase, it is shown how to make guidance commands using these parameters to compensate latitude errors (in lateral channel) and to tune flight time (in longitudinal channel). Also, the capability of these parameters comparing with corresponding parameters in plane geometry to use in guidance algorithm is shown.

2. Length and angle of shortest trajectory on the ellipsoid [8]

Calculation of length and angle of shortest trajectory between two waypoints having their latitude and longitude is known as indirect geodesy problem. In this section, one of the proposed answer to this problem on the ellipsoid is mentioned.

On the shortest trajectory on ellipsoid earth, the constant c is defined as below.

$$c = \sin(\alpha)\cos(\beta) \qquad (1)$$

In this relation α is trajectory angle from north and β is reduced latitude.

Step 1: Consider an initial value for parameter c.

Step 2: Calculate constant k and parameters θ_I and θ_J as below.

$$k = \sqrt{\frac{1-c^2}{1-c^2e^2}}, \theta_I = \sin^{-1}\left(\frac{\sin(\varphi_I)}{k}\right), \theta_J = \sin^{-1}\left(\frac{\sin(\varphi_J)}{k}\right) \qquad (2)$$

In this relation e is eccentricity and, φ_I and φ_J are latitude of initial and final points.

Step 3: Calculate parameter $\Delta\lambda_{calc}$ as below.

$$\Delta\lambda_{calc} = \frac{c(1-e^2)}{\sqrt{1-c^2e^2}} \times \int_{\theta_I}^{\theta_J} \frac{1}{(1-k^2\sin^2\theta)\sqrt{1-k^2e^2 sin^2\theta}} d\theta \qquad (3)$$

Step 4: if there is difference between $\Delta\lambda_{calc}$ and $\lambda_J - \lambda_I$, modify parameter c with a modification coefficient on the difference and repeat steps 2 and 3 until this difference become lower than 10^{-6}rad.

Step 5: Calculate the length of shortest trajectory and trajectory angle from north in initial and final waypoint as below:

$$s_{IJ} = \frac{a(1-e^2)}{\sqrt{1-c^2e^2}} \int_{\theta_I}^{\theta_J} \frac{1}{(1-k^2\sin^2\theta)^{3/2}} d\theta \qquad (4)$$

$$\alpha_I = \sin^{-1}\left(\frac{c}{\cos\beta_I}\right), \alpha_J = \sin^{-1}\left(\frac{c}{\cos\beta_J}\right) \qquad (5)$$

So, to obtain an answer with specified solution, it is needed to repeat first stage up to fourth stage several times. Large formula and high processing reduce the capacity of online implementation of equations to obtain UAV distance and motion angle with respect to a specified point.

For example, consider two points with latitude and longitude $(35^o N, 51^o E)$ and $(40^o N, 117^o E)$. Consider 0.7 as initial value of c. After doing first to third steps, the modifying function of c is implemented as $c = c - 0.1(\Delta\lambda_{calc} - \Delta\lambda_{real})$ and, with tolerance 10^{-6} rad, first step up to third step has been repeated six times. Finally, the value of c becomes 0.73445 and trajectory length becomes 5,728,340 m and angle from north at the first waypoint becomes 63.59^o.

3. Spherical trigonometry

Consider two points A and B on a sphere. To move from A to B, several arcs can be chosen. Two popular arcs are Rhumb line and Great circle. Rhumb line is a trajectory that while moving on it, the motion trajectory angle with meridian lines remain constant. This trajectory that can be shown as a direct line in plane maps, was useful in ancient motion systems that use only few instruments like compass. Despite the appearance of this line in map, it is different from shortest trajectory between two points on the sphere (Fig. 1). The shortest motion trajectory on a sphere is on a concentric circle with sphere that is known as Great circle. According to the expressed characteristics, Great circles have optimum time and fuel, especially in long trajectory. The angle of trajectory respect to the north on different points of great circle changes, so the calculation of desired motion angle and guidance parameters will be hard.

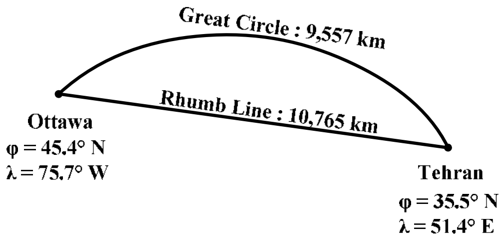


Fig. 1 Rhumb line and Great circle between two points on the map

Consider a spherical triangle that its sides is an arc on great circle like **Error! Reference source not found.**. Each of these sides or arcs have an angle from center of circle. There are relations between anglesα_1,α_2 and α_3 and spherical triangle sides β_1, β_2 and $\beta_3$like sine, cosine and cotangent law, sides cosine law, angles cosine law. Some of these relations that are used in this article, is mentioned.

$$\frac{\sin(\alpha_1)}{\sin(\beta_1)} = \frac{\sin(\alpha_2)}{\sin(\beta_2)} = \frac{\sin(\alpha_3)}{\sin(\beta_3)} \quad (a)$$
$$\cos(\beta_2) = \cos(\beta_1)\cos(\beta_3) + \sin(\beta_1)\sin(\beta_3)\cos(\alpha_2) \quad (b)$$
$$\cos(\alpha_3) = -\cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)\cos(\beta_3) \quad (c)$$
$$\sin(\beta_2)\cos(\alpha_3) = \cos(\beta_3)\sin(\beta_1) - \sin(\beta_3)\cos(\beta_1)\cos(\alpha_2) \quad (d)$$
$$\sin(\alpha_2)\cot(\alpha_1) = \cot(\beta_1)\sin(\beta_3) - \cos(\beta_3)\cos(\alpha_2) \quad (e) \qquad (6)$$

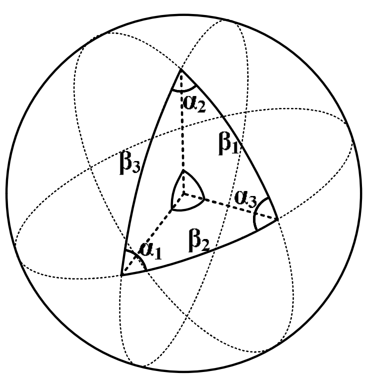


Fig. 2spherical triangle

If one of the angles like α_3 be 90 degree, formulas will be simpler and for example equations **Error! Reference source not found.** and **Error! Reference source not found.** will be:

$$\sin(\beta_1) = \sin(\alpha_1)\sin(\beta_3) \quad (a)$$
$$\cot(\alpha_2) = \tan(\alpha_1)\cos(\beta_3) \quad (b) \qquad (7)$$

To use spherical trigonometry on the spherical earth, it is needed to use Geocentric Latitude instead of geographic latitude. In this article, the purpose of latitude is geocentric latitude. When the object position on the earth is defined by longitude (λ) and latitude (φ), spherical triangle on the earth can be defined by three heads: point A with coordinate(λ_A, φ_A), point B with coordinate(λ_B, φ_B) and North Pole P_N (**Error! Reference source not found.**). In this case, formula of the length of great circle between A and B (S_{AB}) with replacing corresponding parameters in **Error! Reference source not found.** and arc angle to the north in A (γ_{AB}) using equation **Error! Reference source not found.** will be:

$$s_{AB} = \cos^{-1}\big(\sin(\varphi_A)\sin(\varphi_B) + \cos(\varphi_A)\cos(\varphi_B)\cos(\lambda_B - \lambda_A)\big) \quad (a)$$
$$\gamma_{AB} = \tan^{-1}\left(\frac{\sin(\lambda_B - \lambda_A)}{\cos(\varphi_A)\tan(\varphi_B) - \sin(\varphi_A)\cos(\lambda_B - \lambda_A)}\right) \quad (b) \qquad (1)$$

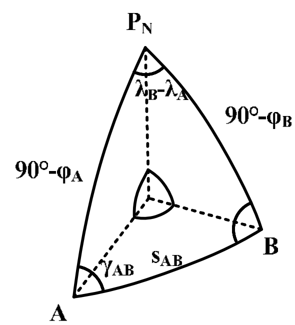


Fig. 3 a spherical triangle on the earth

4. Guidance latitude and longitude

In UAV movement on a desired path from point A to point B, two parameters are important in introducing UAV guidance command, cross track error and distance from points on the desired trajectory. In UAVs and planes, desired trajectory is passing through waypoints with specific latitude and longitude and, a conventional trajectory is great angle between them. In this section spherical trigonometry is used to determine perpendicular distance to desired great circle and passed distance of desired trajectory.

4-1- Determination a pole for great circle between waypoints

In each great circle of the earth, except equator, two vertices is defined that are nearest points of great circle to poles. The angle of great circle with meridian at these vertices will be 90 degree. On equator all points have this characteristic. For IJ great circle (great circle passing through waypoints I and J), these points are v_1 and v_2 in **Error! Reference source not found.**.

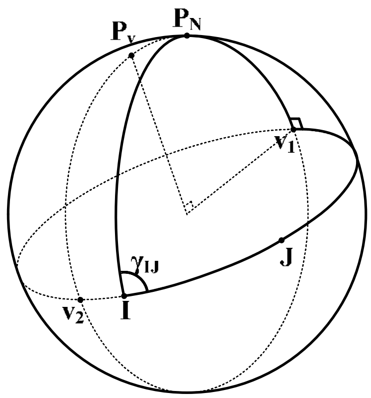


Fig. 4vertices of great circle and its pole

Latitude and longitude of v_1 in **Error! Reference source not found.** for spherical triangle can be obtained using equation **Error! Reference source not found.** with three vertices I, $v_1$andP_N.

$$\varphi_{v_1} = \sin^{-1}\sqrt{1-\sin^2(\gamma_{IJ})\cos^2(\varphi_I)}$$
$$\lambda_{v_1} = \lambda_I + \cot^{-1}\left(\tan(\gamma_{IJ})\sin(\varphi_I)\right) \qquad (9)$$

Meridians are great circles that are perpendicular to equator on each of its points. If we move 90 degree on each meridian from equator, we will reach to poles. These concepts and definitions can be generalized to each great circle on the earth like equator.

So, the great angle that passes through v_1 and is perpendicular to IJ great circle is both conventional meridian and the meridian of IJ great circle. In other word, North Pole (P_N) and the pole of IJ great circle (P_V) are placed on this great circle. So, the coordinate of P_V; regarding it is in north hemisphere or in south hemisphere; will be obtained as below.

$$\begin{cases} \begin{cases} \varphi_{P_v} = \dfrac{\pi}{2} - \varphi_{v_1} \\ \lambda_{P_v} = \pi - \lambda_{v_1} \end{cases} \varphi_{v_1} > 0 \\ \begin{cases} \varphi_{P_v} = \dfrac{\pi}{2} + \varphi_{v_1} \\ \lambda_{P_v} = \lambda_{v_1} \end{cases} \varphi_{v_1} < 0 \end{cases} \qquad (10)$$

4-2- Determination of guidance latitude and longitude of IJ great circle

In the **Error! Reference source not found.** β_1, β_2, β_3 and α_2 are angles of sides and one of vertices of triangle that its vertices are I, P_N and current position (C). At the first stage, these parameters are calculated.

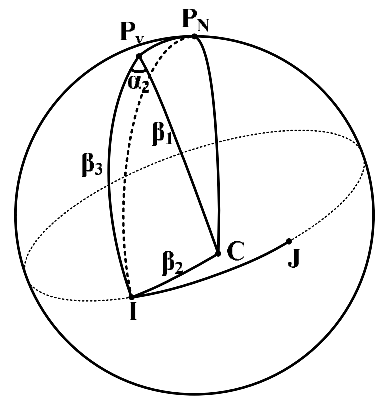


Fig. 5spherical triangles with vertices of I, C, P_N, P_V

The angle $\beta_3$according to pole definition is 90 degree. The angle β_2 with having latitude and longitude of current position of UAV(λ_C,φ_C), in triangle with vertices of I, C, P_N is calculate by formula **Error! Reference source not found.**.

$$\beta_2 = \cos^{-1}\left(\sin(\varphi_I)\sin(\varphi_C) + \cos(\varphi_I)\cos(\varphi_C)\cos(\lambda_C-\lambda_I)\right) \qquad (11)$$

Also, for angle β_1 based on similar formula in triangle with vertices of P_V, C and P_N we have:

$$\beta_1 = \cos^{-1}\left(\sin(\varphi_{P_v})\sin(\varphi_C) + \cos(\varphi_{P_v})\cos(\varphi_C)\cos(\lambda_C-\lambda_{P_v})\right) \qquad (12)$$

With having the angles of β_1, β_2 and β_3, angle α_2 with respect to formula **Error! Reference source not found.** will be obtained.

$$\alpha_2 = \cos^{-1}\left(\frac{\cos(\beta_2)}{\sin(\beta_1)}\right) \qquad (2)$$

So, the parameters λ_C^G and φ_C^G as guidance latitude and longitude of current position of UAV is defined as below.

$$\begin{cases} \lambda_C^G = \alpha_2 \\ \varphi_C^G = \dfrac{\pi}{2} - \beta_1 \end{cases} \qquad (14)$$

With this definition, the geometric concept of guidance latitude and longitude will be the length of arcs that shown in **Error! Reference source not found.**.

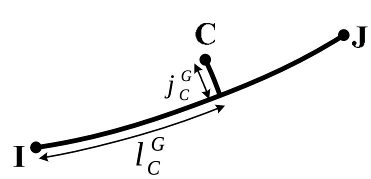


Fig. 6guidance latitude and longitude

By multiplication of guidance latitude and longitude in earth radius, the meaningful parameters that are distance from IJ great circle and distance from point I on the desired trajectory will be obtained. First parameter is used for UAV guidance in lateral channel to keep it on the desired trajectory. The second parameter have an important role in UAV guidance in longitudinal channel to adjust the time of reaching to trajectory waypoints.

5. Guidance algorithm in lateral channel

The role of UAV guidance algorithm in this channel is to minimize cross track error (XTE) or to keep the UAV on the arc between waypoints in horizontal plane. So, no compensation command is needed, if it is placed on the desired trajectory. If it became deviated from desired trajectory, the compensation command can be proportional to XTE. The model of flight kinematic from lateral acceleration to XTE has two poles in origin [11]. So, a proportional controller for generation of lateral acceleration command can cause instability.

A proper guidance algorithm in lateral channel for UAV, is PD algorithm on the XTE for generating lateral acceleration commands [11,12]. The block diagram of UAV guidance loop include guidance algorithm, autopilot dynamic and also flight kinematic is shown in **Error! Reference source not found.**.

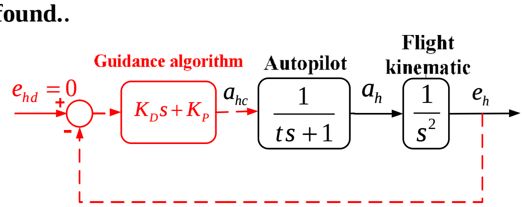


Fig. 7UAV guidance algorithm in lateral channel

In this figure,e_h is the XTE and is equal to$R_e\varphi_C^G$. In this relation R_e is earth radius in UAV position.

6. Guidance algorithm in longitudinal channel

In longitudinal channel, the considered UAV has a velocity control loop and its mission is the adjustment of UAV horizontal velocity to desired value. In this loop, with considering the difference of current velocity from desired velocity, a command for increasing or decreasing the fuel flow and changing the thrust will be send. In this section, with assuming proper performance of velocity control loop, we will analyze time control loop in an upper layer.

Suppose that the time of travelling from waypoint I to J is considered ast_{IJ}. Also, the length of IJ great circle regarding spherical trigonometric relation is calculated as$R_e\lambda_{IJ}$. If we show the flight time of passage from waypoint I as t, a way for generating desired velocity command is momentary determination of it according to remained time and distance to waypoint J. It is done as below.

$$v_{hd} = \frac{R_e(\lambda_{IJ} - \lambda_C^G)}{t_{IJ} - t} \quad (15)$$

In this way, a specific distance error has different impact at the end or at the start of the trajectory and has different changes in desired velocity that is sent to velocity control loop. These changes might be high around the next waypoint and will cause overshoot in velocity control loop and commands that are send to fuel flow control and, it can damage motor seriously.

In the introduced way in this section, first an average speed for arc IJ is determined as below.

$$v_{av} = \frac{R_e\lambda_{IJ}}{t_{IJ}} \quad (16)$$

During movement on the trajectory, according to the time and calculated average velocity, desired guidance longitude is calculated as below.

$$\lambda_d^G = \frac{v_{av}t}{R_e} \quad (17)$$

Now, proportional to the difference of momentary guidance longitude and desired guidance longitude, a modification on desired velocity that is send to velocity control loop is done. **Error! Reference source not found.** shows the control loop of time of travelling from waypoint I to J.

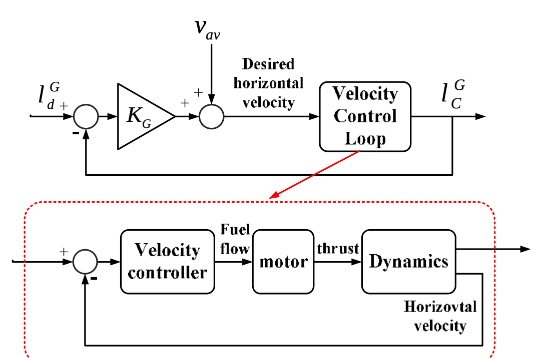


Fig. 8flight time control loop

It is seen that in lateral channel guidance loop, guidance latitude has main role and in longitudinal channel guidance longitude is useful parameter.

7. Results of comparing simulation

In this section, results of implementation of a guidance system in lateral channel and longitudinal channel based on introduced latitude and longitude in comparison with results of corresponding parameters in plane geometry is shown. To do it, consider initial waypoint with coordinates $(j_I, l_I) = (35^{\circ}N, 51^{\circ}E)$ and final waypoint with coordinate $(j_J, l_J) = (40^{\circ}N, 117^{\circ}E)$. The average radius of earth in this latitude is 6,370,212 m. The length of this arc according to trigonometric calculation is 5714400 m and its angle from north in waypoint I will be 63.57º. It is seen that there is very little difference between these calculations and results of repeating integral calculation on ellipsoid. Considered time for passing this trajectory is 400 min that will cause the average horizontal velocity of 238.1 m/s. For this great circle, the coordinate of vertex point will be $(j_v, l_v) = (42.82^{\circ}N, 91.91^{\circ}E)$ and its corresponding pole will be obtained $(j_{PV}, l_{PV}) = (47.18^{\circ}N, 88.09^{\circ}W)$

Although UAV has not cross track error when the motion from waypoint I, track angle error start is 10º. Also, initial horizontal velocity is considered 10 m/s lower than average velocity. In lateral channel, time constant of autopilot of lateral acceleration command is considered 0.3 sec and coefficient K_P and K_D are considered 0.44 and 1.07. In longitudinal channel, the time constant of velocity control loop is 10 sec and K_G is $0.2R_e$.

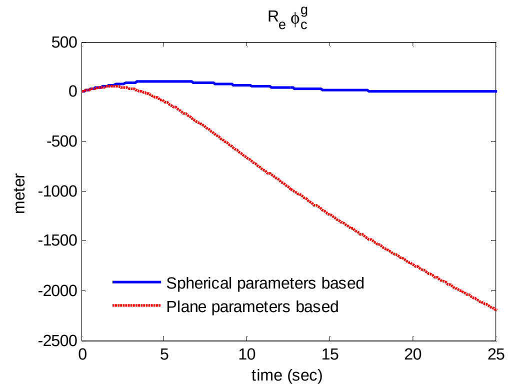
In this case, first guidance simulation is done based on the error of guidance latitude and longitude and then is done based on corresponding parameters in plane geometry that has been used in [13] and can be obtained using below formulas.

$$D = \sqrt{(\varphi_C - \varphi_I)^2 + (\lambda_C - \lambda_I)^2 \cos^2\left(\frac{\varphi_I + \varphi_C}{2}\right)}$$

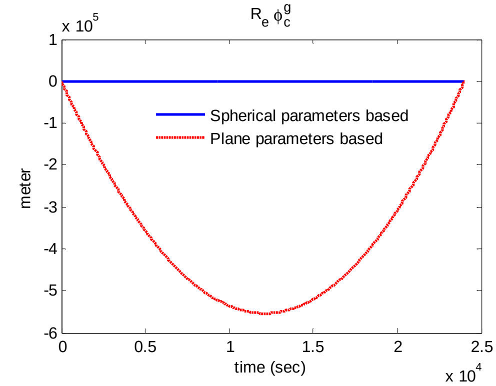
$$\eta = \tan^{-1}\left(\frac{\varphi_C - \varphi_I}{(\lambda_C - \lambda_I)\cos\big(0.5(\varphi_I + \varphi_C)\big)}\right) - \tan^{-1}\left(\frac{\varphi_J - \varphi_I}{(\lambda_J - \lambda_I)\cos\big(0.5(\varphi_I + \varphi_J)\big)}\right)$$

$$\dot{\varphi}_{G-plane} = D\sin(\eta),\ \ \dot{\lambda}_{G-plane} = D\cos(\eta) \qquad (3)$$

The error of lateral channel as $R_e\varphi_C^G$ and the error of longitudinal channel as $R_e(\lambda_d^G - \lambda_c^G)$ in both simulations are shown in **Error! Reference source not found.** and **Error! Reference source not found.**. According to figures, after 20sec in lateral channel, placement on the desired arc is done and after about 90 sec in longitudinal channel, difference from predefined distance is compensated. Also, in the simulation that is done based on plane geometry in comparison with the simulation that is done based on the spherical geometry parameters, maximum value of deviation from desired great circle is about 550 km.

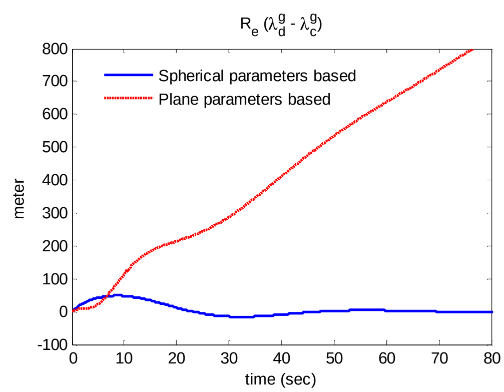


(a)

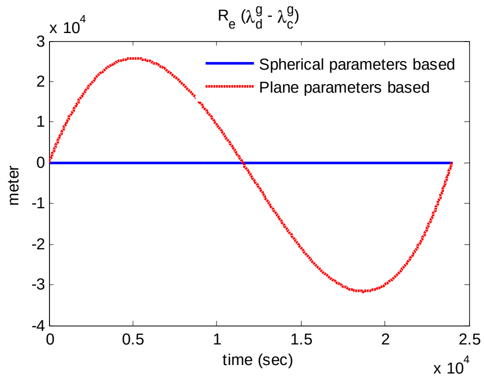


(b)

Fig. 9lateral channel error in placing on the leg in a) initial moments and b) total time of moving cross the leg



(a)



(b)

Fig. 10longitudinal channel error in meeting desired passed distance in a) initial moments and b) total time of moving cross the leg

Also, in Fig. 11 trajectory of aircraft and in Fig. 12**Error! Reference source not found.** changes of lateral acceleration and in Fig. 13 changes of UAV horizontal velocity for two simulations are shown. Although Figure 11 apparently shows that spherical trajectory is longer than plane trajectory, the path length of spherical based trajectory is 133,675m shorter than plane based trajectory. Furthermore average velocity of spherical simulation is 5.57 m/sec lower than average velocity in plane simulation that results in reduction in fuel consumption. Because path angle at the start of trajectory differs in two simulations, the maximum value of lateral acceleration varies. In transient behavior, UAV velocity and lateral acceleration settle in v_{av} after 90 seconds and zero after 25 seconds, consequently.

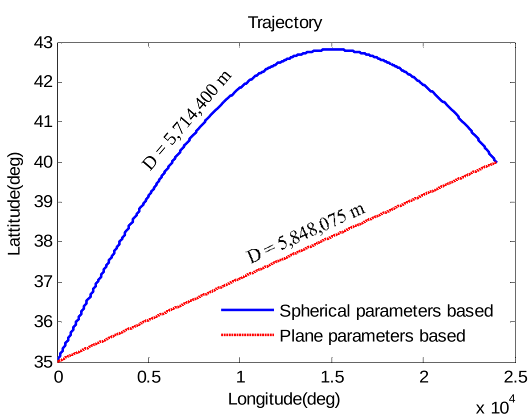


Fig. 11 Trajectory of UAV

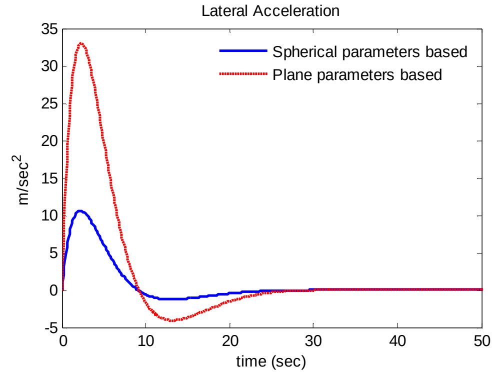


Fig. 12lateral acceleration of UAV

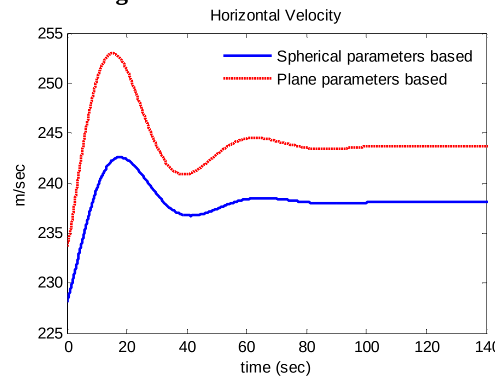


Fig. 13horizontal velocity of UAV

8. Conclusion

In the present article, for crossing trajectory waypoints by UAV, a guidance structure is proposed. Guidance error terms in this structure are defined on the basis of spherical trigonometry. In other words, in this paper it is shown that by using spherical trigonometry we can accurately calculate guidance terms as closed form and without needing to solve complicated integrals. First term is guidance latitude that is proportional to cross track error from great circle between waypoints and second term is guidance longitude that is proportional to arc distance from previous waypoint. Next a PD controller on guidance latitude as guidance algorithm in lateral channel and a proportional controller on guidance longitude as guidance algorithm in longitudinal channel are designed. Finally, these algorithms based on defined parameters are simulated and the performance in comparison with the same algorithms based on the corresponding parameters in plane trigonometry, is shown.

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