Gain Scheduling Observer Based Control Design with Guaranteed Stability and H∞ Performance Consideration

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Abstract

This paper proposes a new method of gain scheduling control design for a nonlinear system which is described as linear parameter varying form. A performance measure based on Linear Matrix Inequality is introduced. To consider stability and performance measures in design process, the H∞ loop-shaping method is used to design the local controllers, which can be described as state feedback observer based structure. By introducing the stability and performance covering condition for the linear parameter varying system, a new interpolation law is proposed, and it is proofed that the resultant controller can preserve the performance measure for the observer based structure for all values of the scheduling parameter. Also the closed loop stability is guaranteed. The method is successfully applied on the control of a well-known benchmark system, namely, the autopilot for a pitch-axis model of an air vehicle. The performance and effectiveness is evaluated against disturbances and parameter uncertainties using computer simulation.

Keywords: gain scheduling; linear parameter varying; observer based controller; stability; performance;

1. Introduction

Gain scheduling has been used successfully to control nonlinear systems for many years and in many different applications [1].Two approach for the synthesis of gain-scheduled controllers exist: the linear parameter varying (LPV) based one and the linearization-based one [1]. In the LPV-based method the controller is synthesized for the nonlinear plant via reformulating it as a linear time varying model. The linearization-based method uses linear time invariant (LTI) models based on Jacobian linearization of the nonlinear plant about a family of equilibrium points. This yields a parameterized family of linearized plants. Then, a linear controller is designed for each region which should guarantees robust stability and performance in the region. Finally, the controller coefficients are changed according to physical parameters which are measured in real time. Using the physical parameters or scheduling variables, the operating region is detected at each time. The controller may be updated via interpolation of certain parameters or switching [2]. It should be noted that in practice, switching among controllers may cause instability of the closed-loop system [3]. Unstable modes and degraded performance may come from the switching transition dynamics, which are not contained in the information provided by each linear model. Closed loop instability may be overcome by imposing a certain dwell time [4, 5]. However, this cannot prevent the undesirable transients.

Guarantees of stability and performance in the whole operating envelope can be obtained using linear parameter varying (LPV) systems theory [6, 7, 8] , but there is no guarantee that a gain scheduled controller which meets the demands be found moreover computational efforts needed to obtain an LPV controller limits its use to low-order and medium-order systems.

In many fields, such as aerospace, there is a strong interest in using the linearization-based gain-scheduling method [9] .For controllers designed independently for each operating point, previous results have been focused on stability [10, 11] or controller switching instead [12, 13]. Particularly, in [14], Youla parameterization has been used, but a network of controllers has been produced which significantly increases the order of the resulting gain-scheduled controler. Some recent results consider the performance problems by establishing an adequate controller initial

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condition when switching [15] or by injecting stabilizing signals among the local controllers, based on bumpless and antiwindup transfer compensators [16]. Bianchi in [17] focuses on formulating a stability-preserving interpolation scheme with a performance level guarantee in the state-space framework, but this leads to high computational complexity and implementation problems.

Observer-based linear parameter-varying control with guaranteed L_2 -gain and H_2 -type performance objectives has been studied by Hakan Koroglu [6]. There are no results that have focused on both stability and H_∞ performance, based on the selection of the observer-based controller for interpolation.

Observer-based controllers are interesting for different practical reasons. The key advantage of these controllers structure comes from the ease of implementation of observer-based controllers. In addition to plant data, only two static gains define the entire controller dynamics. This facilitates the construction of gain-scheduled or interpolated controllers. Indeed, assuming the plants model is available in real-time, observer-based controllers will only require the storage of two static gains to update the controller dynamics at each sample data. Another well-appreciated advantage of these controller structures lie in the fact that the controller states are meaningful variables that estimate the physical plant states. Therefore the controller states can be used to monitor the performance of system, on-line or off-line [18].

This study focuses on stability-preserving interpolation scheme with improved H_∞ performance. This structure does not impose any strong restrictions. In [18] some practical techniques to compute observer-based controller forms to arbitrary compensators have been investigated.

This paper presented a methodology for interpolation of linear time-invariant (LTI) controllers with observer-based form. These controllers are designed for different operating points of a nonlinear system, then interpolated in such a way that produce a gain-scheduled controller with stability and performance guarantees at intermediate interpolation points. In addition, conditions are presented that establish local stability of the nonlinear closed-loop system

The reminder of this paper is organized as follows: Section 2 presents the problem statement، a new gain-scheduled control design is described in Section 3 and the results are illustrated in a pitch-axis air vehicle model in Section 4. Finally, concluding remarks are presented in Section 5.

2. PROBLEM STATEMENT

$$\begin{aligned} \dot{x} &= f(x,u,w) \\ z &= (x,u,w) \\ y &= h(x,u) \end{aligned} \qquad (1)$$

Consider the nonlinear plant Where x is the state, u is the input, ζ denoted an error signal, and y denotes a measured output available to the controllers. w denotes exogenous inputs to the plant such as reference commands, disturbances and noises. Suppose there exist continuous functions, x^o, u^o, w^0 such that for all $r \in \Gamma \subset R^l$,

$$f(x^o(r), u^o(r), w^o(r)) = 0 \qquad (2)$$

Where Γ is a connected compact subset of R^l. For each ρ, the Jacobian linearization of the nonlinear plant (1) about $x^o(\rho), u^o(\rho), w^o(\rho)$ is written as:

$$G(r): \begin{aligned} \dot{x}_d(t) &= A(r)x_d(t) + B_w(r)w_d(t) + B_u(r)u_d(t) \\ z(t) &= C_z(r)x_d(t) + D_z(r)w_d(t) + D_{uz}(r)u_d(t) \\ y(t) &= C_y(r)x_d(t) + D_y w_d(t) \end{aligned} \qquad (3)$$

Where $x_\delta = x - x^o$, $u_\delta = u - u^o$, $w_\delta = w - w^o$, It is assumed that $A_w(\rho), B_w(\rho), C_w(\rho)$, etc. are continues functions on Γ. Suppose that there exists a stabilizing observer based controller (4) designed beforehand and independently at each designing points $\rho_i \in \Gamma, i = 1, \dots, q$. Furthermore, the controller achieves certain H_∞ performance specifications, γ_i, $i = 1, \dots q$.

$$C(r_i): \begin{aligned} \dot{\hat{x}}(t) &= (A(r_i) + B_u(r_i)K_i + L_i C_y(r_i))\hat{x}(t) - L_i y(t) \\ u_i(t) &= K_i x(t) \end{aligned} \qquad (4)$$

Then the objective is to formulate a new interpolation scheme for the state feedback gains and state observer gains such that:

i. The gain-scheduled controller (5) Stabilizes the plant $G(\rho)$ defined in (3) at any point $\rho \in \Gamma$.

ii. The gain-scheduled controller (5) guarantees an H_∞ norm bound constraint on disturbance attenuation for all $\rho \in \Gamma$.

Furthermore, to find sufficient conditions to ensure local stability of the nonlinear closed-loop system.

$$C(r): \begin{array}{l} \dot{\hat{x}}(t) = (A(r) + B_u(r)K(r) + L(r)C_y(r))\hat{x}(t) - L(r)y(t) \\ u(t) = K(r)x(t) \end{array} \qquad (5)$$

3. GAIN-SCHEDULED CONTROLLER SYNTHESIS

First, a preliminary result on sufficient conditions such that an observer based controller guarantees H_∞ norm bound constraint on disturbance attenuation is presented. Then the results on stability and performance preserving interpolation of observer based controllers for a fixed linear plant are extended to parameter-varying plants.

Lemma 1

Suppose that observer based controller (6) stabilizes the LTI plant (7).

$$\begin{array}{l} \dot{\hat{x}}(t) = (A + B_u K + LC_y)\hat{x}(t) - Ly(t) \\ u(t) = Kx(t) \end{array} \qquad (6)$$

$$G: \begin{array}{l} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t) \\ z(t) = C_z x(t) + D_z w(t) + D_{uz} u(t) \\ y(t) = C_y x(t) + D_y w(t) \end{array} \qquad (7)$$

Furthermore, assume there exist symmetric, positive-definite matrices P, Q such that

$$\begin{bmatrix} P(A+B_uK)^T + (A+B_uK)P & -B_uK & B_w & P(C_z + D_{uz}K)^T \\ -(B_uK)^T & (A+LC_y)^T Q + Q(A+LC_y) & Q(B_w + LD_y) & -(D_{uz}K)^T \\ B_w^T & (B_w + LD_y)^T Q & -gI & D_z^T \\ (C_z + D_{uz}K)P & -D_{uz}K & D_z & -gI \end{bmatrix} < 0 \qquad (8)$$

Then the closed-loop system (9) satisfies $sup \frac{\|\zeta_{(t)}\|_2}{\|w_{(t)}\|_2} = \|G_{K,L}\|_\infty \le \gamma$

$$G_{K,L}: \begin{array}{l} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & B_uK \\ -LC_y & A + B_uK + LC_y \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B_w \\ -LD_y \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} C_z & D_{uz}K \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + D_z w(t) \end{array} \qquad (9)$$

In the other words, the observer based controller (6) guarantees the H_∞ norm bound constraint on disturbance attenuation.

Proof

First, we rewritten the closed-loop system (9) as follows,

$$G_{K,L}: \begin{array}{l} \underline{\dot{x}}' = A_{cl}\underline{x}' + B_{cl}\underline{w} \\ z = C_{cl}\underline{x}' + D_{cl}\underline{w} \end{array} \qquad \underline{x}' = \begin{bmatrix} x(t) \\ e(t) = x(t) - \hat{x}(t) \end{bmatrix} \qquad (10)$$

$$A_{cl} = \begin{bmatrix} A + B_uK & -B_uK \\ 0 & A + LC_y \end{bmatrix} \qquad B_{cl} = \begin{bmatrix} B_w \\ B_w + LD_y \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} C_z + D_{uz}K & -D_{uz}K \end{bmatrix} \qquad D_{cl} = D_z$$

Equation (8) can be written as (11),

$$\left[\begin{array}{c|cc} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} (A_{cl}^T \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} + \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} A_{cl}) \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} B_{cl} & \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} C_{cl}^T \\ \hline B_{cl}^T \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} & -gI & D_{cl}^T \\ C_{cl} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} & D_{cl} & -gI \end{array}\right] < 0 \qquad (11)$$

Again, we can rewrite (11) as (12),

$$\begin{bmatrix} R & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \left[\begin{array}{c|cc} A_{cl}^T W + WA_{cl} & WB_{cl} & C_{cl}^T \\ \hline B_{cl}^T W & -gI & D_{cl}^T \\ C_{cl} & D_{cl} & -gI \end{array}\right] \begin{bmatrix} R & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \qquad (12)$$

Where

$$R = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} > 0 \quad , \quad W = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} > 0 \qquad (13)$$

It is clear that R, W are symmetric, positive-definite matrices. Using bounded real lemma and (12), yield:

$$\left[\begin{array}{c|cc} A_{cl}^T W + WA_{cl} & WB_{cl} & C_{cl}^T \\ \hline B_{cl}^T W & -gI & D_{cl}^T \\ C_{cl} & D_{cl} & -gI \end{array}\right] < 0 \quad \Rightarrow \quad \sup_w \frac{\|z(t)\|_2}{\|w(t)\|_2} = \|G_{K,L}\|_\infty \le g \qquad (14)$$

Lemma 2

Suppose that the observer based controllers (15) both stabilize the LTI plant (16).

$$\begin{array}{l} \dot{\hat{x}}(t) = (A + B_u K_i + L_i C_y)\hat{x}(t) - L_i y(t) \\ u_i(t) = K_i x(t) \end{array} \qquad i = 1, 2 \quad (15)$$

$$G: \begin{array}{l} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t) \\ z(t) = C_z x(t) + D_z w(t) + D_{uz} u(t) \\ y(t) = C_y x(t) + D_y w(t) \end{array} \qquad (16)$$

Furthermore, assume there exist symmetric, positive-definite matrices Q_1, Q_2 and $P_1 = P_2$ such that,

$$S_i=\begin{bmatrix} P_i\,(A+B_uK_i)^T+(A+B_uK_i)P_i & -B_uK_i & B_w & P_i\,(C_z+D_{uz}K_i)^T \\ -(B_uK_i)^T & (A+LC_y)^TQ_i+Q_i(A+LC_y) & Q_i(B_w+L_iD_y) & -(D_{uz}K_i)^T \\ B_w^T & (B_w+L_iD_y)^TQ_i & -g_iI & D_z^T \\ (C_z+D_{uz}K_i)P_i & -D_{uz}K_i & D_z & -g_iI \end{bmatrix}<0 \qquad (17)$$

$$for\quad i=1,2$$

Then for any fixed m, $0\le m\le 1$, the observer based controller (19) stabilize (16) with,

$$\sup_w \frac{\|z(t)\|_2}{\|w(t)\|_2}=\|G_{K,L}\|_\infty\le g_m \qquad \min(g_1,g_2)\le g_m\le\max(g_1,g_2)$$

$$\dot{\hat{x}}(t)=(A+B_uK+LC_y)\hat{x}(t)-Ly(t)$$
$$u(t)=Kx(t)$$

$$K=(mK_1P_1+(1-m)K_2P_2)P^{-1}\quad,\qquad P=mP_1+(1-m)P_2$$
$$L=Q^{-1}(mQ_1L_1+(1-m)Q_2L_2)\quad,\qquad Q=mQ_1+(1-m)Q_2$$

where $G_{K,L}$ is the closed-loop system.

$$G_{K,L}:\begin{cases}\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix}=\begin{bmatrix}A & B_uK\\ -LC_y & A+B_uK+LC_y\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}+\begin{bmatrix}B_w\\ -LD_y\end{bmatrix}w(t)\\ z(t)=\begin{bmatrix}C_z & D_{uz}K\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}+D_zw(t)\end{cases}$$

Proof

Lemma 1 and (19) yield, $\|G_{K_1,L_1}\|_\infty\le\gamma_1$ and $\|G_{K_2,L_2}\|_\infty\le\gamma_2$. Evaluating $mS_1+(1-m)S_2$ for fixed m, $0\le m\le 1$ yields after some computations,

$$\begin{bmatrix} P\,(A+B_uK)^T+(A+B_uK)P & -B_uK & B_w & P\,(C_z^T+D_{uz}K)^T \\ -(B_uK)^T & (A+LC_y)^TQ+Q(A+LC_y) & Q(B_w+LD_y) & -(D_{uz}K)^T \\ B_w^T & (B_w+LD_y)^TQ & -g_mI & D_z^T \\ (C_z+D_{uz}K)P & -D_{uz}K & D_z & -g_mI \end{bmatrix}<0$$

Where

$$K=(mK_1P_1+(1-m)K_2P_2)P^{-1}\quad,\qquad P=mP_1+(1-m)P_2$$
$$L=Q^{-1}(mQ_1L_1+(1-m)Q_2L_2)\quad,\qquad Q=mQ_1+(1-m)Q_2$$

$$g_m=mg_1+(1-m)g_2$$

Lemma 1, (22) and (23) result, $\|G_{K,L}\|_\infty\le g_m$ and $\min(g_1,g_2)\le g_m\le\max(g_1,g_2)$

In [19], the stability covering condition is introduced for stable interpolating between state feedback gains, we generalized this for observer based controllers. It is noted that A_{cl} is system matrix and we say a matrix is stable if it has negative real-part eigenvalues.

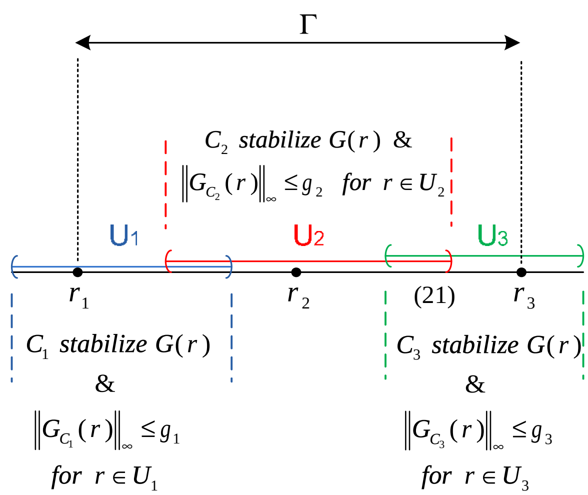
Definition 1 (17)

Generalized stability covering condition. Suppose that each arbitrary controller C_i stabilizes (3) at designing points $\rho_i\in\Gamma, i=1,\dots,q$. In the other words, $A_{cl}(\rho_i)$ is stable, $i=1,\dots,q$. Let U_i,containing ρ_i, be an open neighborhood such (18) that $A_{cl}(\rho_i)$ is stable for each fixed $\rho\in U_i$. If $\Gamma\subset\cup_{i=1}^{q}U_i$ then we say that the controllers satisfy the generalized stability covering condition.

Definition 2

(19)

Stability-performance covering condition. Assume that all conditions of Definition 1 hold. In addition, each arbitrary controller C_i satisfies $\sup_w\frac{\|\zeta\|_2}{\|w\|_2}=\|G_{C_i}(\rho)\|_\infty\le\gamma_i$ for $\rho\in U_i$, $i=1,\dots,q$, then we say that the controllers satisfy the stability-performance covering condition. Figure 1 illustrates how the controllers C_1,C_2,C_3 satisfy stability-performance covering condition. (20)



(22)

Figure 1. C_1,C_2,C_3 satisfy Stability-performance covering condition. $\Gamma\subset\bigcup_{i=1}^{3}U_i$ Theorem 1

(23)

Consider the linear parameter varying plants (3), which is the Jacobian linearization of the nonlinear plant (1) about equilibrium $x^o(\rho),u^o(\rho),w^o(\rho)$, $\rho\in\Gamma\subset R$, Assume that there exists a stabilizing observer based controller (4) designed beforehand and independently at each designing points $\rho_i\in\Gamma, i=1,\dots,q$, $(\rho_1<\dots<\rho_q)$. Suppose also that the controllers satisfy stability-performance covering condition,($\Gamma \subset \bigcup_{i=1}^{q} U_i$). If there exists $\upsilon > 1$ and symmetric positive-definite matrices $Q_1, \ldots, Q_q, P_1 = \cdots = P_q$ Such that

$$\begin{bmatrix} P_i(*)+(A(r)+B_u(r)K_i)P_i & -B_u(r)K_i & B_w(r) & P_i(C_z(r)+D_{uz}(r)K_i)^T \\ * & (*)Q_i+Q_i(A(r)+L_iC_y(r)) & Q_i(B_w(r)+L_iD_y(r)) & -(D_{uz}(r)K_i)^T \\ * & * & -g_iI & D_z^T(r) \\ * & * & * & -g_iI \end{bmatrix} < -uI \qquad (24)$$

$$for \quad r \in U_i, i = 1, \ldots, q$$

Then there exist intervals $[b_i, c_i] \subset (U_i \cap U_{i+1} \cap [\rho_i, \rho_{i+1}]), i = 1, \ldots, q-1$, such that the observer based gain-scheduled controller ***(25)*** Stabilizes the plant $G(\rho)$ defined in (3) at any fixed point $\rho \in \Gamma$ and $\sup_w \frac{\|\zeta\|_2}{\|w\|_2} = \|G_C(\rho)\|_\infty \le \max(\gamma_1, \ldots, \gamma_q)$, $\rho \in \Gamma \subset R$. Furthermore, the linear parameter varying system ***(27)*** is exponentially stable if $\rho(t)$ satisfies (28).

$$C(r): \begin{array}{l} \dot{\hat{x}}(t) = (A(r)+B_u(r)K(r)+L(r)C_y(r))\hat{x}(t) - L(r)y(t) \\ u(t) = K(r)x(t) \end{array}$$

Where

$$K(r) = \begin{cases} K_i & r \in [r_i, b_i) \\ \bar{K}_i(r)P^{-1}(r) & r \in [b_i, c_i] \\ K_{i+1} & r \in (c_i, r_{i+1}] \end{cases}, \quad L(r) = \begin{cases} L_i & r \in [r_i, b_i) \\ Q^{-1}(r)\bar{L}_i(r) & r \in [b_i, c_i] \\ L_{i+1} & r \in (c_i, r_{i+1}] \end{cases}$$

$$\bar{K}_i(r) = \frac{c_i - r}{c_i - b_i}K_iP_i + \frac{r - b_i}{c_i - b_i}K_{i+1}P_{i+1}$$

$$\bar{L}_i(r) = \frac{c_i - r}{c_i - b_i}Q_iL_i + \frac{r - b_i}{c_i - b_i}Q_{i+1}L_{i+1} \qquad i = 1, \ldots, q-1$$

$$P(r) = \begin{cases} P_i & r \in [r_i, b_i) \\ \frac{c_i - r}{c_i - b_i}P_i + \frac{r - b_i}{c_i - b_i}P_{i+1} & r \in [b_i, c_i] \\ P_{i+1} & r \in (c_i, r_{i+1}] \end{cases} \qquad (26)$$

$$Q(r) = \begin{cases} Q_i & r \in [r_i, b_i) \\ \frac{c_i - r}{c_i - b_i}Q_i + \frac{r - b_i}{c_i - b_i}Q_{i+1} & r \in [b_i, c_i] \\ Q_{i+1} & r \in (c_i, r_{i+1}] \end{cases}$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A(r) & B_u(r)K(r) \\ -L(r)C_y(r) & A(r)+B_u(r)K(r)+L(r)C_y(r) \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \qquad (27)$$

$$|\dot{r}(t)| < \min_{i=1,\ldots,q-1} \frac{|c_i - b_i|}{\|Q_{i+1} - Q_i\|} \qquad t \ge 0 \qquad (28)$$

Proof

Since stability-performance covering condition is satisfied, there exist intervals $[b_i, c_i] \subset (U_i \cap U_{i+1} \cap [\rho_i, \rho_{i+1}]), i = 1, \ldots, q-1$.

A. For $r \in [r_i, b_i) \subset U_i, i = 1, \ldots, q-1$ using Lemma 1 and (24) yield $C_i(r)$ stabilize $G(r)$ and $\|G_{C_i}(r)\|_\infty \le g_i$.

B. For $r \in (c_i, r_{i+1}] \subset U_{i+1}, i = 1, \ldots, q-1$ using Lemma 1 and (24) yield $C_i(r)$ stabilize $G(r)$ and $\|G_{C_{i+1}}(r)\|_\infty \le g_{i+1}$.

C. For $r \in [b_i, c_i] \subset U_i, U_{i+1}, i = 1, \ldots, q-1$ we define (24) as follows:

$$S_j = \ldots$$

$$\begin{bmatrix} P_j(*) + (A(r) + B_u(r)K_j)P_j & -B_u(r)K_j & B_w(r) & P_j(C_z(r) + D_{uz}(r)K_j)^T \\ * & (*)Q_j + Q_j(A(r) + L_jC_y(r)) & Q_j(B_w(r) + L_jD_y(r)) & -(D_{uz}(r)K_j)^T \\ * & * & -g_jI & D_z^T(r) \\ * & * & * & -g_jI \end{bmatrix} < 0 \quad (29)$$

$$for \quad r \in [b_j, c_j], \quad j = i, i+1 \qquad i = 1, \ldots, q-1$$

$m(r)$ is defined as follows:

$$m(r) := \frac{c_i - r}{c_i - b_i} \quad (30)$$

$$\begin{bmatrix} P(*) + (A(r) + B_u(r)K)P & -B_u(r)K & B_w(r) & P(C_z(r) + D_{uz}(r)K)^T \\ * & (*)Q + Q(A(r) + LC_y(r)) & Q(B_w(r) + LD_y(r)) & -(D_{uz}(r)K)^T \\ * & * & -g_{m_i}I & D_z^T(r) \\ * & * & * & -g_{m_i}I \end{bmatrix} < -uI \quad (31)$$

$$r \in [b_i, c_i] \quad , i = 1, \ldots, q-1$$

Where

$$K(r) = \bar{K}_i(r)P^{-1}(r) \quad , \quad L(r) = Q^{-1}(r)\bar{L}_i(r)$$

$$\bar{K}_i(r) = \frac{c_i - r}{c_i - b_i}K_iP_i + \frac{r - b_i}{c_i - b_i}K_{i+1}P_{i+1}$$

$$\bar{L}_i(r) = \frac{c_i - r}{c_i - b_i}Q_iL_i + \frac{r - b_i}{c_i - b_i}Q_{i+1}L_{i+1} \qquad r \in [b_i, c_i] \quad , i = 1, \ldots, q-1$$

$$P(r) = \frac{c_i - r}{c_i - b_i}P_i + \frac{r - b_i}{c_i - b_i}P_{i+1}$$

$$Q(r) = \frac{c_i - r}{c_i - b_i}Q_i + \frac{r - b_i}{c_i - b_i}Q_{i+1}$$

$$g_{m_i} = mg_i + (1-m)g_{i+1}$$

So for $r \in [b_i, c_i]$, $i = 1, \ldots, q-1$ the observer based controller C(r) stabilize $G(r)$ and $\left\|G_{C_i}(r)\right\|_\infty \le g_{m_i}$. It is clear that $g_{m_i} \le \max(g_i, g_{i+1})$.

Since $\Gamma \subseteq \bigcup_{i=1}^{q} U_i$ and according to results of A, B and C yield,

$$\left\|G_C(r)\right\|_\infty \le \max(g_1, \ldots, g_q) \qquad for \quad r \in \Gamma$$

Proof for locally exponential stability of parameter varying system (27) is similar to that presented in [19]. By defining a change of coordinates

$$\frac{d}{dt}V(z,t) = z^T \begin{bmatrix} A_K(r) + \frac{d}{dt}P^{-1}(r) & -eP^{-1}(r)B_u(r)K(r) \\ -eK^T(r)B_u^T(r)P^{-1}(r) & A_L(r) + \frac{d}{dt}Q(r) \end{bmatrix} z \le -d_3\|z\|^2 \quad (39)$$

Where

Evaluating $mS_i + (1-m)S_{i+1}$ for fixed m, $0 \le m(r) \le 1$ yields after some computations,

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad T = \begin{bmatrix} eI & 0 \\ I & -I \end{bmatrix}$$

With e a real constant, exponential stability of (27) is implied by exponential stability of (35).

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} A(r) + B_u(r)K(r) & -eB_u(r)K(r) \\ 0 & A(r) + L(r)C_y(r) \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \quad (32)$$

For given $r(t)$, The Lyapunov function (36) is choosen.

$$V(z,t) = V_1(z_1,t) + V_2(z_2,t) = z_1^TP^{-1}(r(t))z_1 + z_2^TQ(r(t))z_2$$

Where symmetric positive-definite matrices $P(r)$ and $Q(r)$ are from (26). For $i = 1, 2,$ there exist $d_{1i}, d_{2i}, d_{3i} > 0$ such that

$$d_{1i}\|z_i\|^2 \le V_i(z,t) \le d_{2i}\|z_i\|^2, \qquad \frac{d}{dt}V_i(z_i,t) \le -d_{3i}\|z_i\|^2$$

and

$$\min\{d_{11}, d_{12}\}\|z\|^2 \le V(z,t) \le 2\max\{d_{21}, d_{22}\}\|z\|^2 \quad (33)$$

For e sufficiently small and $0 < d_3 < \min\{d_{31}, d_{32}\}$,

$$A_K(r) = (A(r) + B_u(r)K(r))^T P^{-1}(r) + P^{-1}(r)(A(r) + B_u(r)K(r))$$
$$A_L(r) = (A(r) + L(r)C_y(r))^T Q(r) + Q(r)(A(r) + L(r)C_y(r)) \tag{40}$$

By using Lemma 4.2.8 in [20], (27) is exponentially stable if $r(t)$ satisfies (41),

$$|\dot{r}(t)| < \min\left\{ \min_{i=1,\dots,q-1} \frac{|c_i - b_i|}{\|P_{i+1} - P_i\|},\ \min_{i=1,\dots,q-1} \frac{|c_i - b_i|}{\|Q_{i+1} - Q_i\|} \right\} \qquad t \ge 0 \tag{41}$$

Since, it is assumed $P_{i+1} = P_i$ then $|\dot{r}(t)| < \min_{i=1,\dots,q-1} \frac{|c_i - b_i|}{\|Q_{i+1} - Q_i\|}$ is sufficient to guarantee exponential stability of (27).

4. EXAMPLE

To illustrate the procedure of the interpolation technique, design of a gain scheduled autopilot for a pitch-axis air vehicle model is considered. Nonlinear model, performance and robust stability requirements are founded in [9], the model is valid for $1.5 \le Mach \le 3$ and $-20deg \le \alpha \le +20deg$. The nonlinear system is converted to the parameter variable form. Plant equilibrium families are parameterized by the Mach$(\Gamma: 1.5 \le Mach \le 3)$.Then the appropriate operating points are determined using the algorithm proposed in [9]. The robust linear control by H_∞ loop shaping method is designed in such a way that the design criteria, including step tracking, maximum of overshoot, settling time, etc. are met, and the loop shaped system has a robust stability margin of $\varepsilon_i = \gamma_i^{-1}, i = 1, \dots, q$ in each operating point. Then the designed robust controllers are implemented in observer-based structure. The characteristics of the step response are listed in Table **1**. It is clear that the designed robust controller has truly fulfilled the expected performance characteristics in every operating point. The characteristics related to the robust stability margin in the specified operating points are given in Table **2**.

Table 1. The characteristics of the step response in each designing point

Operating Point	**[α,M]**	**Rise Time (90%)**	**Settling Time (2%)**	**Over Shoot**	**Under Shoot**
P1	[5,1.5]	0.43s	0.87s	0	7.7%
P2	[5,1.95]	0.38s	0.71s	0	8.5%
P3	[5,2.55]	0.33s	0.56s	0.02%	8.9%
P4	[5,3]	0.31s	0.52s	0.07%	8.1%

Table 2. The characteristics related to the robust stability margin in each designing point

Operating Point	**[α,M]**	**Robust Stability Margin (ε)**	**Performance Level $(\gamma = \varepsilon^{-1})$**	**GM(dB)**	**PM(deg)**
P1	[5,1.5]	0.564	1.77	15.1	74.8
P2	[5,1.95]	0.578	1.73	14.9	75.1
P3	[5,2.55]	0.554	1.81	14	73.4
P4	[5,3]	0.551	1.81	13.7	73.7

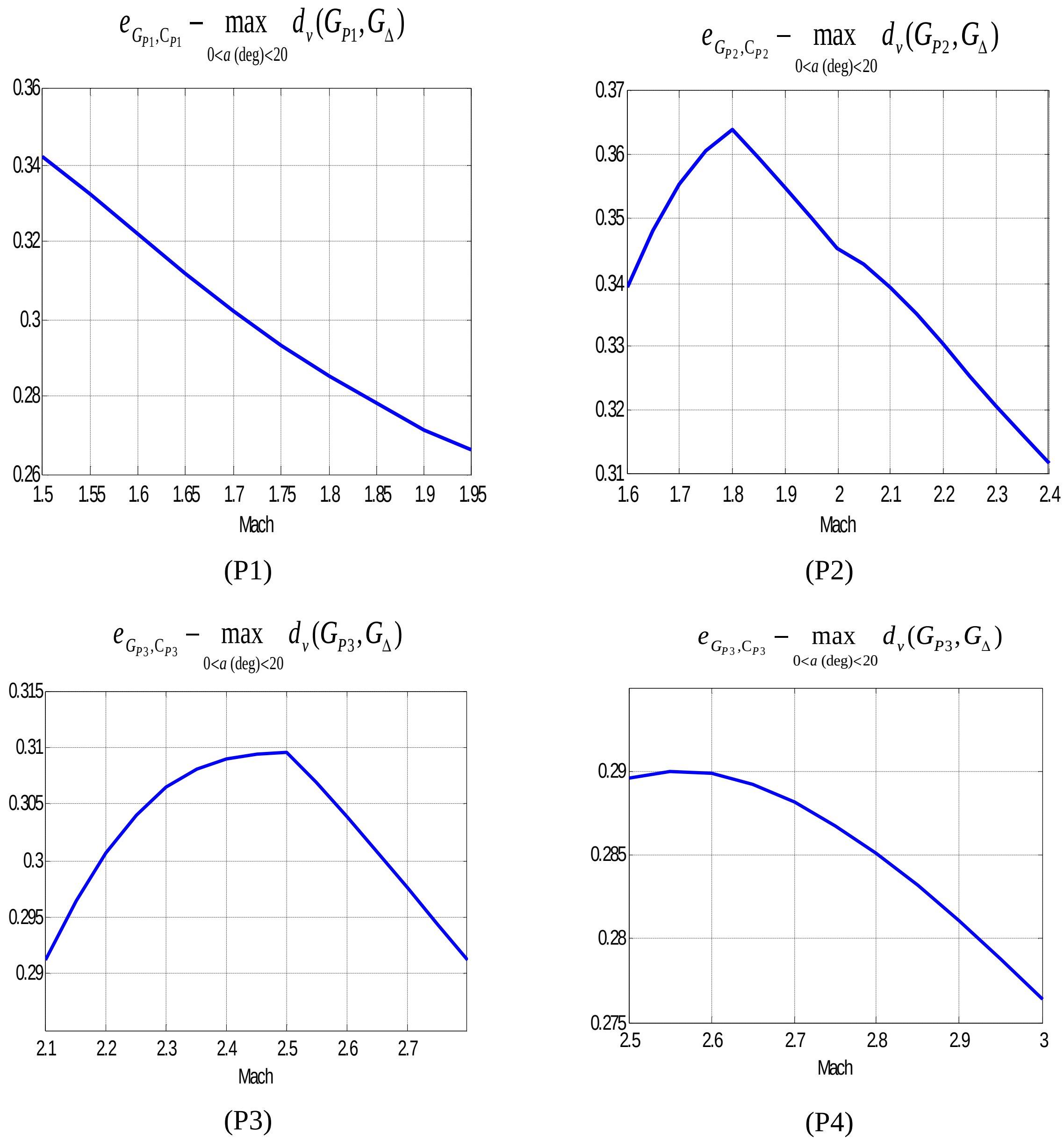


Figure 2. The lower limit of the robust stability margin that the designed controllers in the points of P_1, P_2, P_3, P_4 can make for their neighborhoods

A study on the stability-performance covering conditions

By applying some theorems about the v-gap metric in [21], it will be shown that the designed local controllers will satisfy the stability-performance conditions. From Figure 2 it is evident in the whole workspace, we have $\varepsilon_{G_0,C(\rho)} - \delta_v(G_0, G_\Delta) > 0$. In other words, in the entire workspace $\delta_v(G_0, G_\Delta)$ is less than $\varepsilon_{G_0,C(\rho)}$, so from [21] the designed controllers in the operating points of P_1, P_2, P_3, P_4 guarantee the stability of closed loop system for any fixed operation point in the entire workspace. The lower limit of the robust stability margin that the designed controllers in the points of P_1, P_2, P_3, P_4 can make for their neighborhoods are illustrated in Figure 2.Regarding this figure, one can obtain U_i $(i = 1, \ldots, 4)$ as well as γ_i $(i = 1, \ldots, 4)$,

It is evident from the

Table 3 that $\Gamma \subseteq \bigcup_{i=1}^{4} U_i$ (Γ: $1.5 \le Mach \le 3$). Thus the local controllers meet the stability-performance covering conditions,

Figure 3).

Regarding the method of interpolation between the controllers, it is required that the overlapping regions of U_is be determined, and the interpolation been carried out inside these overlapping regions. In the following, the overlapping regions of U_is between every two nominal operating points are given.

$$[b_i \quad c_i] \subset U_i \cap U_{i+1} \qquad i = 1,2,3$$

$$U_1 \cap U_2 = [1.6 \quad 1.8]$$
$$U_2 \cap U_3 = [2.1 \quad 2.2]$$
$$U_3 \cap U_4 = [2.7 \quad 2.8]$$

Table 3. Interval U_i, lower limit of robust stability margin and performance level in the neighborhoods of each designing point P_i, (U_i).

Ui	Lower limit of robust stability margin (ε)	Performance Level ($\gamma = \varepsilon^{-1}$)
$U_1 = [1.5, 1.8]$	0.28	$\gamma_1 = 3.6$
$U_2 = [1.6, 2.2]$	0.33	$\gamma_2 = 3.0$
$U_3 = [2.1, 2.8]$	0.29	$\gamma_3 = 3.4$
$U_4 = [2.7, 3.0]$	0.27	$\gamma_4 = 3.7$

For non-common areas the local controller structure, and for common areas of stability-performance, i.e. $U_i \cap U_{i+1}$ the method of interpolation stated in Theorem 1 is used. The common areas are shown in Table **4**.

Table 4. The overlapping regions of U_is between every two nominal operating points

i	$U_i \cap U_{i+1}$	$[b_i \ c_i] \subseteq U_i \cap U_{i+1}$	b_i	c_i
1	[1.6 1.8]	[1.6 1.8]	1.6	1.8
2	[2.1 2.2]	[2.1 2.2]	2.1	2.2
3	[2.7 2.8]	[2.7 2.8]	2.7	2.8

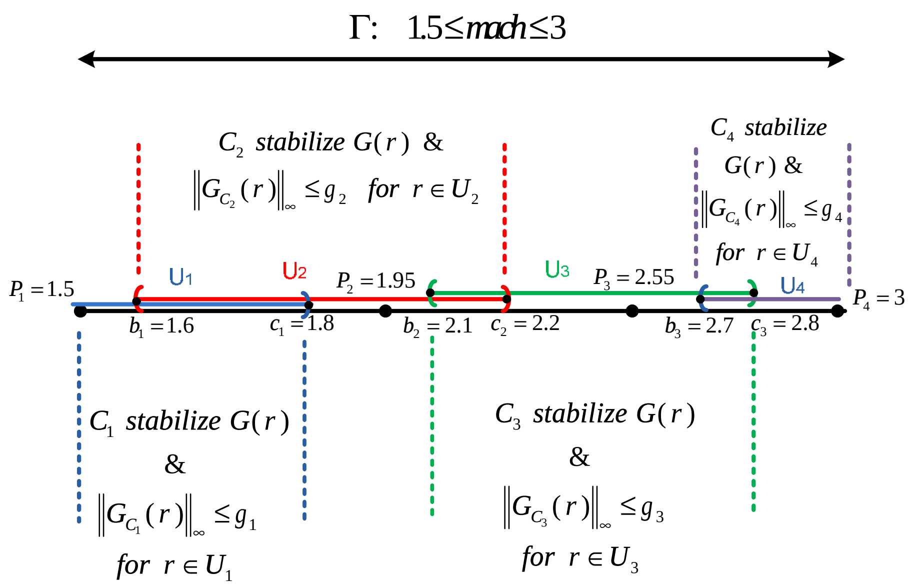


Figure 3. The local controllers $C_{i'}(i = 1, \dots, 4)$ meet the stability-performance covering conditions

Nonlinear simulation

After designing the controllers in the operating points and presenting the method of interpolation between the controllers, it is time to simulate the nonlinear behavior of the designed autopilot. As can be seen in Figure 4, in the nonlinear simulation the air vehicle tracks the acceleration commands with a settling time less than 1 second, almost zero steady state error and less than 10% overshoot. ***Figure 6*** shows the changes in the angle of attack of the flight vehicle in this scenario, which has covered almost the entire workspace of the object, i.e. -20° to 20°. It is worth noting that the mathematical model of the air vehicle is itself a function of Mach and angle of attack. But in this autopilot, only the Mach was selected as the scheduling variable, which means that the changes of the angle of attack does not affect the controller gains. However, proper selection of the operating points in the controller design as well as utilizing the method of H_∞ loop shaping in designing the controller so that in each Mach the system remains robust against the changes in the angle of attack, resulted in the fact that by choosing only one scheduling variable, all the desired design characteristics (stability and performance) are met. In addition using this new method of interpolation guarantees the preserving of a criterion of the system's performance during interpolation as well as its stability.

The mathematical model of the flying object is practically derived through some tests in the wind tunnel and calculating its aerodynamic coefficients. Therefore there exist some uncertainties in the mathematical model of the flying object, especially in the aerodynamic coefficients of its torque. In the studied air vehicle, there exists about 25% uncertainty around the nominal values in the aerodynamic coefficients. The response of the flight vehicle to independent changes of these coefficients are shown in ***Figure 8***. As can be seen, the designed autopilot has properly maintained its stability and performance against the parametric uncertainties.

Wind is often one of the disturbances that affect air vehicles. This disturbance is "output disturbance" and directly affects the angle of attack of the flight vehicles. The wind disturbance has been modeled in the form of a step with 5° amplitude, and is applied in sec. 3. ***Figure 9*** shows the effect of this disturbance on the response of the flying object. As can be seen the effect of this disturbance has rejected in less than 1 second.

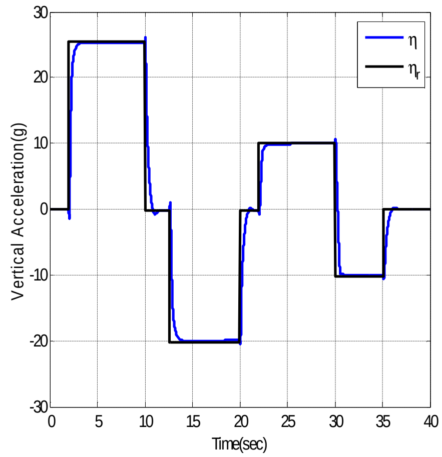


Figure 4. Acceleration commands and air vehicle esponse.

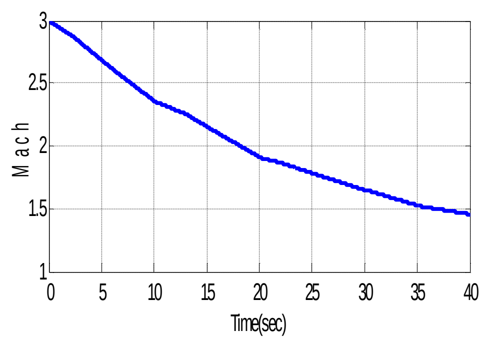


Figure 5. The changes in the Mach

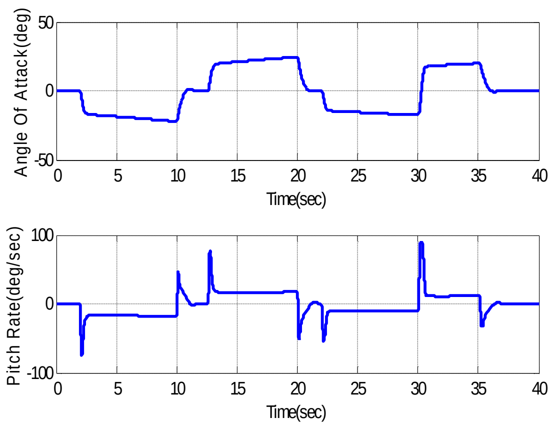


Figure 6. Air vehicle states

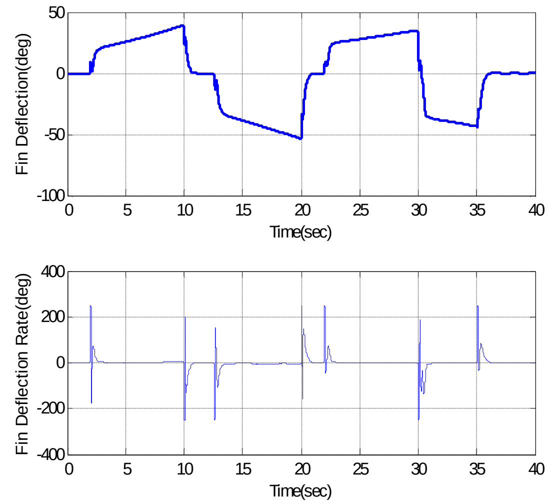


Figure 7. Fin deflection and its derivative.

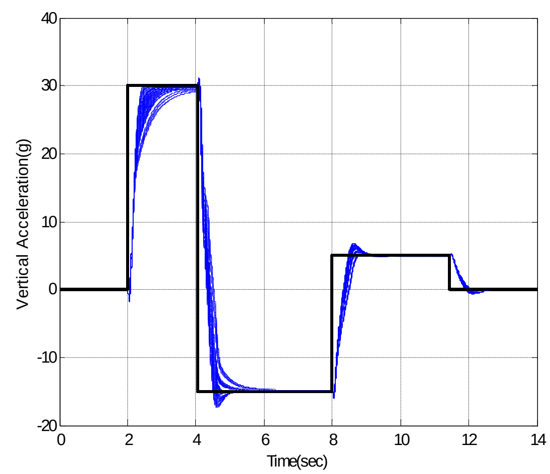


Figure 8. Parametric robustness of the controller.

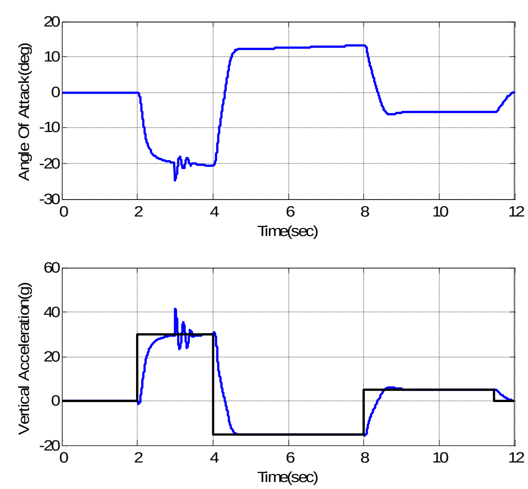


Figure 9. Acceleration commands and air vehicle response in presence of wind disturbance

5. CONCLUSION

This paper presented a methodology for interpolation of linear time-invariant controllers with observer-based form. These controllers are designed for different operating points of a nonlinear system, then, interpolated in such a way that produce a gain-scheduled controller with stability and performance guarantees at intermediate interpolation points. In addition, conditions are presented that establish local stability of the nonlinear closed-loop system. As mentioned in the introduction, observer-based structure of controllers does not impose any strong restrictions because some practical techniques for determining the observer-based controller form for any compensator with arbitrary order have been investigated. A well-appreciated advantage of this structure is implementation viewpoint. The method was successfully applied on the control of a well-known benchmark system, namely, the pitch-axis model of an air vehicle.

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